AN INTRODUCTION TO THE THEORY OF ELECTRICITY.

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TO THE

THEORY OF ELECTRICITY,

WITH NUMEROUS EXAMPLES.

BY

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PREFACE.

THE present work has grown out of an attempt, while giving my Cheltenham pupils the experimental details of electricity, to impart at the same time to the more intelligent among them consistent though elementary ideas of the theory which underlies the experiments. At the instance of friends interested in scientific teaching, I undertook to prepare from the notes I had used for my class the work which now issues from the press. Among such friends I must specially refer to my former colleague, J. A. Fleming, Esq. B. Sc., who rendered me considerable assistance in arranging the plan of the earlier portions of the work.

Geometrical as distinguished from analytical methods have been employed, and although a large proportion of the propositions involve the ideas of the Doctrine of Limits, the use of the notation of the Calculus has been avoided. A competent knowledge of Calculus is rare among schoolboys, and experience as a teacher has shown that a geometrical investigation often gives a grasp of the method where an analytical one would give only the result.

The foundation of the method employed is really the conception of Lines of Force, so largely used by Faraday in his researches as a means of exhibiting without mathematical symbols the quantitative relations of a field of force; relations, which assume at once a numerical expression by help of Prof. Stokes' beautiful theorem given in the 7th and 8th Propositions of the second Chapter.

I must take this opportunity of acknowledging the debt I (in common with all modern students of electricity) owe to the writings of Sir W. Thomson, Prof. Clerk Maxwell, and M. Wiedemann, which I have consulted at every step. In addition to these, I have derived profit from a large number of miscellaneous papers. References to results obtained from these may have through oversight been omitted, and for such omissions I must crave indulgence.

I have also to record my personal thanks to Rev. T. N. Hutchinson of Rugby, and to W. J. Lewis, Esq. of the British Museum, who have read through the whole work in course of preparation, and have afforded me throughout many valuable suggestions and criticisms.

My special thanks are also due to my friend and former pupil, H. W. Reynolds, Esq., I.C.S., who has devoted a large amount of valuable time to correcting with me the proofs and testing the examples. I owe largely to his care and accuracy such measure of freedom as I may enjoy from typographical and other errors. I can hardly hope to have escaped from such accidents entirely, and shall be thankful to students who will send me a note of any which occur in the course of their reading.

LINNÆUS CUMMING.

RUGBY, *November* 1, 1876.

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CHAPTER I.

PHYSICAL UNITS.

1. THE measurement of all physical quantities depends ultimately on the units of *space, time*, and *mass*.

In England the units are generally the *foot, second*, and *pound;* but we shall adopt the centimetre, gramme, and second (C. G. S.) system, which is now almost universally used for scientific purposes.

2. DEF. I. VELOCITY *is the rate of motion of a body, and if uniform is measured by the number of centimetres passed over per second; if variable it is measured, at any instant, by the number of centimetres which would be passed over per second supposing the velocity uniform during that second, and of the same value as at the instant under consideration.*

3. It will be seen from the above definition, that velocity is a property of a body at any given instant, and not necessarily the same during a finite interval. Thus when we speak of a train going thirty miles an hour, we do not mean to say that it has gone thirty miles in the past hour, or that it will go thirty miles in the next hour; but that supposing the velocity to remain uniform, it would go thirty miles during that time.

4. The unit of velocity is the velocity of a body which goes over one centimetre per second.

If a body moving uniformly with velocity v pass over a space s in time t, the relation between these quantities is clearly expressed by the formula

$$s = vt \quad \text{..............................(i).}$$

5. Def. II. Acceleration *is the rate of change of velocity, and is measured, when uniform, by the number of units of velocity added on to a body's motion per second. When variable it is measured, at any instant, by the number of units of velocity which would be added on per second, supposing the acceleration constant, and of the same value as at the instant under consideration.*

6. Like velocity, acceleration is a property of a body at a particular instant, not necessarily continuing the same through a finite interval. It is the measure of the body's *quickening* at that instant.

7. The unit of acceleration is the acceleration of a body whose velocity increases by a unit of velocity per second. If a body be moving under a uniform acceleration f through a time t, and if V be the initial velocity, and v the velocity at the end of the time t, then

$$v = V + ft \qquad \text{(ii)}.$$

The best illustration of a uniform acceleration, is the motion of a body near the earth's surface. In this case it is proved by experiment that the acceleration due to the earth is represented numerically by 981 (at Paris). Thus a body falling to the earth has its velocity increased each second by 981 centimetres per second. This does not mean that the body describes 981 centimetres in the second, or even describes 981 centimetres less in one second than in the next, but that if, for instance, the body is projected downwards with a velocity of 100 centimetres per second, it will have at the end of the first second a velocity of 1081 centimetres per second, at the end of the second second its velocity will be 2062 centimetres per second, and so on during each second of the motion.

Retardation is treated as negative acceleration. If for instance a body be projected upwards, its velocity is diminished by 981 cm. per second each second, and generally if f represent the retardation, our formula (ii) becomes

$$v = V - ft \qquad \text{(ii)}'.$$

If the resulting velocity should be negative it will denote that the body is moving with a certain velocity in the direction opposite to that of projection.

8. To find the space described during a given time t by a body moving with uniform acceleration, we may consider that since the acceleration is uniform, the *average velocity* during the interval will be the same as the velocity at the middle of the interval, and this will clearly be $V+\frac{1}{2}ft$. The space described will be the same as that due to this velocity during the time t. Hence, by formula (i),

$$s=(V+\tfrac{1}{2}ft)\,t$$
$$=Vt+\tfrac{1}{2}ft^2 \quad\text{.................... (iii)},$$

or if the acceleration is negative,

$$s=Vt-\tfrac{1}{2}ft^2 \quad\text{....................(iii)}'.$$

Combining (ii) and (iii) by algebra, we have from (ii)

$$v=V+ft;$$

or

$$v^2=V^2+2Vft+f^2t^2$$
$$=V^2+2f(Vt+\tfrac{1}{2}ft^2),$$

from (iii)

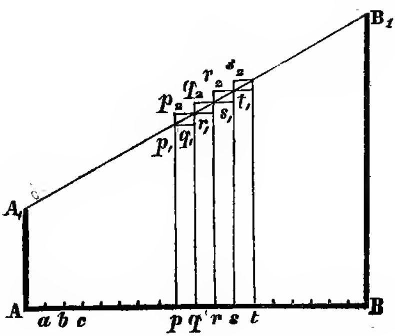
$$=V^2+2fs \quad\text{..............................(iv)}.$$

Similarly, from (ii)′ and (iii)′,

$$v^2=V^2-2fs \quad\text{..........................(iv)}'.$$

9. The space described may also be illustrated graphically by a method which will be of frequent use.

Fig. 1.



Set off along a horizontal line AB equal lengths Aa, ab, bc, &c., representing short intervals into which the whole time AB of the motion of the body can be divided, raise at A, a, b, &c. straight lines (called ordinates) perpendicular to AB, and of such lengths as to represent on a certain scale the velocities of the body at the end of each interval. Let these lines be AA_1, aa_1, $bb_1 \ldots pp_1$, qq_1, $rr_1 \ldots BB_1$, then drawing a system of complete parallelograms p_2q_1, q_2r_1, $r_2s_1 \ldots$ &c.; the space described during the small interval of time pq will be represented numerically by something between $pq \times pp_1$ and $pq \times qq_1$, or by some area between p_1q and p_2q; since qq_1 represents the velocity at the beginning of the interval and pp_1 the velocity at the end. The whole space described will be intermediate between the sum of all the parallelograms p_2q, q_2r, r_2s, &c., and the sum of all the parallelograms p_1q, q_1r, r_1s, &c. The difference of these two sums is clearly a parallelogram whose height is BB_1 and base one of the intervals pq, and therefore equals $BB_1 \times pq$, and if the intervals are sufficiently small this difference is indefinitely small, and each of the sums becomes the same as the whole area A_1ABB_1, and this therefore will represent the whole space described. Since the acceleration is uniform, the increments of velocity are the same for the same increments of time, and consequently by Euc. VI. Prop. ii. the line A_1B_1 is a straight line, also $AA_1 = V$, and $BB_1 = V + ft$, and $AB = t$. Hence the area of the trapezium

$$
\begin{aligned}
&= \tfrac{1}{2}(AA_1 + BB_1)\, AB \\
&= \tfrac{1}{2}(V + V + ft)\, t \\
&= Vt + \tfrac{1}{2}ft^2,
\end{aligned}
$$

which agrees with our formula (iii).

It must be carefully noted that the area A_1ABB_1 has no actual relation to the space described beyond the numerical one here represented. Thus if seconds be represented by centimetres along AB, and units of velocity by centimetres perpendicular to AB, then the number of square centimetres enclosed by the lines AA_1, BB_1, AB, A_1B_1 represents numerically the number of units of space passed over by the body in the time AB.

When in future we make use of this graphical representation of a formula, we shall indicate its construction by saying that *abscissæ*, or distances set off along a horizontal line such as AB, are to be taken to represent the number of units in one magnitude, and *ordinates* or lines perpendicular, to represent some other co-related magnitude; and from the nature of the figure so formed, we shall deduce by geometry various relations.

10. DEF. III. DENSITY. *Matter is that in virtue of which and through which all forces act, and is itself as incapable of definition as space and time. We however require a formula to express the amount of matter in any given volume. This depends on the substance, and the density of a given substance is defined as the amount of matter in a unit of volume.*

Hence if M is the amount of matter in a body, D the density, and V the volume,

$$M = DV \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}.$$

M is generally called the *mass* of the body.

11. Our unit of mass is arbitrary, and can be most conveniently expressed as the unit volume of some standard substance. The unit of volume must be the cube of the linear unit, viz. a cubic centimetre. The standard substance is water at its maximum density, i.e. at a temperature of 4^0 Cent. The unit of mass will thus be a cubic centimetre of distilled water at 4^0 Cent. This mass is called the gramme, and is the basis of the decimal system of weights.

When we speak of a particle of matter, we mean a mass of matter which can be acted on by forces, but which in its geometrical relations can be treated as a point.

12. DEF. IV. FORCE *is defined as that which changes or tends to change a body's state of rest or motion, and any given Force may be measured by the acceleration it imparts to a gramme.*

Thus a unit force is that which imparts to a gramme a unit acceleration, and is called a *dyne*.

13. The science of Physics is founded upon certain experimental truths, which were first given concisely by Newton. They are called the Laws of Motion.

14. LAW I. *Every body if at rest, remains at rest; and if in motion, moves uniformly in a straight line unless acted upon by some extraneous force.*

This is often called the law of Inertia of Matter, expressing that matter has no tendency to move without the application of force. It is impossible to establish it experimentally, as every body in the universe is moving, and subject to a great complexity of forces. We may, however, establish *relative rest*, as of a body resting on a horizontal plane, and we moreover observe not only that it never sets itself in motion, but that when the body is started the smoother the plane the more slowly is the velocity diminished.

15. LAW II. *If any number of forces act on a body at rest, or in motion, each force produces its own effect, both in magnitude and direction, independently of the existing motion.*

This is shown experimentally by dropping a stone from the mast-head of a ship which is moving uniformly. The stone is found to fall at the foot of the mast, both in the same time, and in the same position, as if the ship had been at rest; the pull of gravity on the stone not being interfered with by its own uniform horizontal motion, which was necessarily that of the ship at the moment it was dropped.

16. This law leads to some important results:

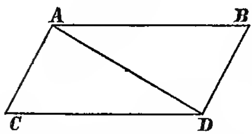
(i) We can prove that if any statical force acts on a body, the measure of the force will be the product of the mass moved into the acceleration under which it is moving. If one unit of force act on a gramme, and if another unit of force act upon it in the same direction, each (by the second law of motion) produces it own effect, and the acceleration is two units. Similarly, if there be three units of force, the acceleration will be three units, and so on.

Hence if f units of force act, there will be an acceleration f imparted to the gramme. Again, conceive a mass of m grammes. Let it be cut up into m separate masses each of one gramme, and let f units of force be applied to each of them. They will move on side by side, each having f units

of acceleration. Let them be now joined so as to form one mass. The *whole force* acting on it now, is mf units of force, and the acceleration produced in the mass m is f. In other words, the measure of *Force* is the product of the mass moved into the acceleration impressed on it, or, as it is usually written,

$$P = mf.$$

Fig. 2.



(ii) *Parallelogram of Velocities**. If a particle of matter placed at A move with a uniform velocity along AB, so that in one second it reaches B, AB is its velocity; if similarly it move with a velocity AC from A along AC, it will be at C at the end of one second. But if these independent velocities were impressed simultaneously, by the second law of motion it would be at the same place at the end of one second, as if it had first been allowed to move for one second along AB, and then for one second along BD parallel and equal to AC. But this would land it at D, where D is the opposite extremity of the diameter of a parallelogram of which AB, AC are adjacent sides.

If we consider the position of the particle at any time during the second, the paths described in directions AB and AC will be proportional to AB and AC, and by similar triangles the particle will be on AD. Hence the particle has passed from A to D along AD in one second, and AD therefore represents the velocity of the particle in magnitude and direction under the combined effect of the two movements.

We shall hereafter allude to this method of obtaining

* It may be objected that the Parallelogram of velocities follows from the definition of velocity, and is not of the nature of a *physical law*. This is true if we view it as presenting only a mental image of the movement of a point, but it becomes a physical law so soon as we assume that *a body* can have two velocities impressed upon it at the same time, and the result of these impressed velocities can only be ascertained by direct experiment. This experiment is implicitly involved in the statement of the second Law of Motion, and in virtue of it the Parallelogram law rises from a geometrical notion to a natural law.

the joint effect of two separate effects as the *Parallelogram Law.*

(iii) *Parallelogram of Forces.* Next let AB, AC represent two accelerations impressed simultaneously on a body. Then AB represents the velocity which the body would have added to it in one second supposing the acceleration AB alone to act. Similarly AC represents the velocity added under the acceleration AC alone. And since we may compound velocities by the Parallelogram law, it is clear that AD represents in magnitude and direction the joint effect of the two accelerations acting simultaneously.

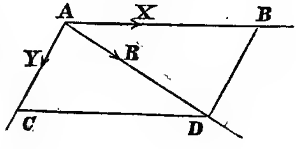
This joint effect of two or more accelerations or velocities is often called the Resultant acceleration or velocity, and the two or more separate velocities or accelerations are called the Components. In the above cases AD is the resultant velocity or acceleration of the two component velocities or accelerations AB, AC.

Since for a given mass the force moving it is proportional to the acceleration imparted to it, the parallelogram law will hold good for combining forces just as for accelerations.

17. Thus we see that accelerations and forces can be compounded two and two by drawing the diagonals of the parallelograms of which they are adjacent sides: also accelerations or forces can be resolved in any two directions by drawing lines parallel to these directions through the extremities of a line representing the acceleration or velocity.

Thus, suppose an acceleration or force R to act along AD (fig. 3), and we require to resolve it along AB, AC, if we set off any length AD, and complete the parallelogram $ABCD$, then it is clear that AB, AC bear the same numerical relation to the components that AD does to R. Hence the components are

Fig. 3.



$$\frac{AB}{AD} R \text{ along } AB = X, \text{ suppose,}$$

and $$\frac{AC}{AD} R \text{ along } AC = Y,$$

or $$\frac{BD}{AD} R \text{ along } AC = Y;$$

where ABD is a triangle whose sides are proportional to the acceleration or force and its components. Similarly, if we draw any other triangle whose sides are parallel to AD, AB, AC, the acceleration and its components will be proportional to its sides also.

If AB, AC be at right angles to each other, and α the angle between AD and AB,

$$X = R\cos\alpha, \quad Y = R\sin\alpha, \quad \frac{Y}{X} = \tan\alpha \ldots\ldots\text{(vi)}.$$

18. Of the nature of Force are all weights, pressures, tensions of strings, attractions and repulsions between bodies.

It will be convenient to express the unit Force in terms of our standard weight. That taken as the ordinary standard is the weight of a gramme at the sea-level in the latitude of Paris. Now it is known by experiment that in this latitude the acceleration of a falling body is 981. Hence the unit of weight is a gramme under 981 units of acceleration. Therefore a unit of Force $= \frac{1}{981}$ of the weight of a gramme.

In future weights will be measured in grammes and converted into absolute units of force by multiplication by 981. If a weight be given as w grammes the measure of it in units of force is $981w$.

19. LAW III. *To every action there is an equal and opposite reaction, whether the bodies so acting and reacting be at rest or in motion.*

This expresses the fact that when a body is pressed, it presses back with an equal force.

If for example I press my finger on the table, the table presses my finger back with the same force with which

I press the table. If a horse tow a boat along a canal the horse is dragged back with exactly the force it uses to drag the boat forward.

20. DEF. V. MOMENTUM *or Quantity of Motion is defined as the product of mass into velocity.*

Since acceleration is the rate of change of velocity, it is clear (Art. 16) that Force may be defined as the rate of change of momentum. By the third law of motion, when bodies act on each other, action and reaction are equal and opposite; or in other words, at every instant the rates of change of momenta are equal and opposite; and since the time from the beginning of the action is the same for both, the whole change of momentum during the action is the same for both; but the directions are opposite, so that the momentum gained by one equals that lost by the other.

Since momentum for a given mass is proportional to velocity, the parallelogram law holds for momenta as well as for velocities. We have then proved the parallelogram law in the case of velocities, accelerations, forces, and momenta.

21. DEF. VI. MOMENT OF A FORCE. *The moment of a force about a given point is defined as the product of the force into the perpendicular from the given point on to the line of action of the force.*

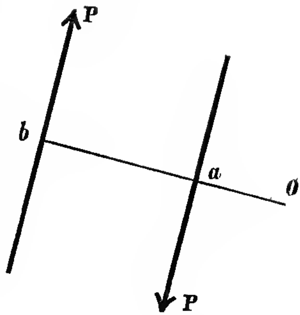
The moment of a force thus defined measures the tendency of a force to turn a body about a given point as axis. And it is obvious that when the forces on a body balance each other, the sum of the tendencies of all forces which twist it in one direction, must exactly balance the tendencies of all forces twisting it in the opposite direction.

22. DEF. VII. COUPLE. *Two forces which are equal in magnitude and parallel, acting in opposite directions, but not in the same straight line, are termed a couple.*

It is clear that, if we take any point O (fig. 4) in the plane of the forces, and from it draw a perpendicular Oab to the two forces, the difference between their moments about O is always $P.Ob-P.Oa=P.ab$. Thus either force multiplied

by the perpendicular distance between the forces is called the *moment of the couple*, and measures the tendency of the couple to twist the body round.

Fig. 4.



From this it follows, that no single force can balance a couple. For if possible suppose it to be balanced by any force. Then choosing O a point in the line of the force, its moment about O vanishes, and there remains the moment $P.ab$ unbalanced.

23. DEF. VIII. WORK *may be defined as resistance overcome through space.*

Thus if a body be carried from one position to another against an accelerating force, such as a weight raised, a certain amount of muscular or other power must be expended. This Work is measured, when the acceleration is uniform, by the mass moved into the acceleration against which it is moved into the space through which it is moved. If W be the work expended in moving a mass m, against an acceleration f, through a space s, we have

$$W = mfs \qquad \text{(viii)}.$$

The unit of work will be done in moving a gramme through a centimetre against a unit of acceleration, and is called an *erg*. The ordinary English unit of work is the foot-pound, being the work done in raising a pound through a foot.

Rate of work is the number of ergs done per second by any machine.

The English standard of working power is usually called the "horse-power," and is defined as 33000 foot-pounds per minute.

To convert this into ergs we have only to remember that

$$1 \text{ centimetre} = \cdot 3937079 \text{ inch},$$
$$1 \text{ gramme} = 15\cdot 43235 \text{ grains}.$$

Hence $$1 \text{ foot} = \frac{12}{\cdot 3937079} \text{ cms.},$$

$$1 \text{ lb. Avoirdupois} = 7000 \text{ grains Troy},$$
$$= \frac{7000}{15\cdot 43235} \text{ gms.}$$

Therefore $$1 \text{ foot-pound} = \frac{12 \times 7000}{15\cdot 43235 \times \cdot 3937079} \text{ cm.-gms.};$$

which, remembering that gravity = 981 units of acceleration, becomes

$$= \frac{12 \times 7000 \times 981}{15\cdot 43235 \times \cdot 3937079} \text{ ergs.}$$

Therefore one horse-power

$$= 33000 \text{ foot-pounds per } 1',$$
$$= \frac{33000}{60} \text{ foot-pounds per } 1'',$$
$$= \frac{33000 \times 12 \times 7000 \times 981}{60 \times 15\cdot 43235 \times \cdot 3937079} \text{ ergs per } 1'',$$
$$= 7460 \text{ million ergs per } 1'' \text{ nearly.}$$

Note. If a body be under several different accelerations in the same direction, the work done against all will clearly be the sum of the work done against each separately through the same space. For if the separate accelerations in direction of motion be $f_1, f_2, f_3, \ldots$, the work done against them in moving a mass m through a space s,

$$= m(f_1 + f_2 + f_3 + \ldots)\, s$$
$$= mf_1 s + mf_2 s + mf_3 s + \ldots$$
= sum of work done against each acceleration separately.

24. DEF. IX. KINETIC ENERGY *or* VIS VIVA *is defined as half the product of the mass into the square of the velocity of a body.*

The importance of this quantity depends on the fact that when a body is in motion, the change in vis viva is always

equal to the work done. We have already proved in formula (iv) Art. 8, that when a particle is moving with a uniform acceleration f,

$$v^2 = V^2 + 2fs.$$

If the mass be m,

$$mv^2 - mV^2 = 2mfs;$$

therefore $$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2 = mfs.$$

Here the left-hand side of the equation represents the change in the vis viva of the body, and the right-hand side the work done in transferring the body from one position to the other.

This is proved above only for a uniform acceleration, but if the acceleration is not uniform we may divide the whole time of the body's motion into very short intervals, during each of which the acceleration may be considered constant, and then the sum of all the changes in vis viva will be equal to the sum of all the work done on the body during each of these intervals. It follows therefore that for all accelerations the change in vis viva is equal to the work done on a body in passing from one position to another.

The same will hold good for any system of particles whatever, since the whole vis viva of the system is the sum of the vis viva of each particle, and the whole work done on the system is the sum of all the work done on each particle.

25. The vis viva measures the power of a body to do work or overcome resistance. This is practically found to be the case, e.g. a bullet which with a certain velocity pierces one plank, will with double that velocity pierce four planks of the same thickness.

26. If we cause by any means a body to move against the action of a force, work is said *to be done upon it*, or *against the force*.

If a body moving under the action of a force overcomes resistance it is said *to do work*.

27. DEF. X. ENERGY *is defined to be capacity for doing work.*

In Article 24 it was proved that when work is *done on*

a body, vis viva is produced in the body, and the amount of work done is numerically equal to the amount of vis viva produced. If on the other hand the vis viva of a body is diminished, either work is done by it, or else the body gains a position in which it has advantage relatively to work, that is, in which it has a latent capacity for doing work. It has then energy stored up in it. This energy is called Potential Energy. When a body is brought under the action of any forces they are said to *do work on the body*. This work may either cause it to do work on other bodies, or it may be converted into the equivalent vis viva.

Energy is therefore of two kinds,

(i) *Kinetic* or *Actual*, when the body is in absolute motion.

(ii) *Potential* or *Latent*, when the body, in virtue of work done upon it occupies a position of advantage, so that the work can be at any time recovered, by the return of the body to its old position.

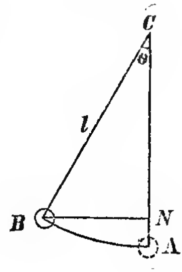
Thus, for example, if I carry a ball to the top of a house I do work upon the ball and give the ball a store of potential energy. If the ball now fall down, as long as it is falling its potential energy is being converted into vis viva, and its vis viva on reaching the ground is exactly the numerical equivalent of the work done in carrying it up. We may now conceive the ball as striking the end of a lever, and being brought gradually to rest. If this lever be in connection with a frictionless crane it would be found that the crane would just raise a weight as heavy as the ball to the height from which the ball had fallen, before its energy was destroyed.

In the first part of the process the ball would be having work done upon it, and in the second it would be doing work.

28. In the case of a pendulum the same principle holds good. The original impact by which the pendulum is started communicates to it vis viva, and the bob comes to rest when the potential energy due to its rise is equivalent to the vis viva imparted.

Thus, if a pendulum bob of mass m be started with a vis viva $\frac{1}{2}mV^2$ at A, and if on reaching B the velocity is destroyed, then, if through B a line BN be drawn horizontally, the bob has been lifted through AN,

Fig. 5.



$$\therefore\ mgAN = \tfrac{1}{2}mV^2,$$

where g denotes the acceleration due to the attraction of the earth on the bob.

But $$AN = l\,(1 - \cos\theta)$$

$$= 2l\sin^2\frac{\theta}{2},$$

where $$l = \text{length of pendulum};$$

$$\therefore\ mg2l\sin^2\frac{\theta}{2} = \tfrac{1}{2}mV^2;$$

$$\therefore\ V^2 = 4lg\sin^2\frac{\theta}{2};$$

$$V = \sin\frac{\theta}{2}\sqrt{4lg},$$

$$mV = m\sqrt{4lg}\sin\frac{\theta}{2}.$$

Hence with a given pendulum the momentum at starting is proportional to the sine of $\frac{1}{4}$ of the whole angle of swing, or to the chord of half the arc of swing.

But mV measures the blow by which the pendulum was started. Hence in any case of an instantaneous force applied to a pendulum, we shall assume the blow is proportional to the sine of half the angle of deflection. This is called the principle of the ballistic pendulum.

The same principle will apply further on to a magnet swinging in a uniform field.

29. The principle illustrated in the two last Articles, that work can only be done at the expense of energy, is of universal application, and will be used largely in application to Electricity. It is known as the principle of 'Conservation of Energy'.

EXAMPLES ON CHAP. I.

The following relations may be assumed:

1 metre = 39·3708 inches.
1 pound avoirdupois = 453·59 grammes.
1 cubic foot of water weighs 1000 oz.

The acceleration of gravity = 32·2 when ft. and sec. are fundamental units.

The abbreviation cm. is used for centimetre.
.... gm. ....gramme.
.... sec. ....second.
.... sq. ....square.
.... cub. ....cubic.
.... den....... ....density.

1. How many centimetres are there in a foot? How many sq. cm. are there in a sq. foot? How many cub. cm. are there in a cub. foot? *Ans.* 30·4794 : 929 : 28315.

2. Express in metrical units the velocity of sound which travels 1100 feet per sec. *Ans.* 33527·7 cm. per sec.

3. How many yards per minute and miles per hour are described by a body which travels at the rate of 1000 cm. per sec.? *Ans.* 656·18 : 22·37.

4. The acceleration of gravity is measured by 981 in the metrical system. Find its numerical value when feet and seconds are employed as fundamental units. *Ans.* 32·2.

5. A stone is allowed to fall from a cliff: with what velocity is it moving at the end of the fourth second?

6. A cannon-ball is shot vertically upwards and ascends for five seconds, then returning back again.

(i) With what velocity was it projected?

(ii) What height did it reach?

(iii) What time elapses between leaving the gun and returning to earth?

(iv) If it was caught at the instant of turning and hurled down with a velocity of 1000 feet per second, what would be its velocity on reaching the ground?

(v) In the last case how long would it take during its fall? *Ans.* $\frac{64}{161}''$.

7. The Moon's distance is 60 times the Earth's radius. Through what distance does the Earth pull the Moon every minute? Assuming that the Moon moves in a circle, and that the radius of the Earth is 4000 miles, calculate the length of the lunar month. *Ans.* 16·1 ft.; 27·3 days.

8. A person drops a stone down a well, and hears the splash after 2·86 sec. Find the depth of the well, making allowance for the time taken by the sound in coming up (see Ex. 2). *Ans.* 121 feet.

9. A balloon ascends vertically and uniformly for 4·5 sec., and a stone is then let fall which takes seven sec. to reach the ground. Find the velocity of the balloon and its height when the stone was dropped.

Ans. 68 ft. per sec.; 306 ft.

10. A stone after falling for one sec. strikes a pane of glass and loses half its velocity. How far will it fall in the next second? *Ans.* ~~32~~ ft.

11. What is the volume in cub. cms. of 1·5 kilogms. of iron whose density is 7·25? *Ans.* 206·9.

12. What is the weight in grammes of 10,000 cub. cms. of sea-water whose density is 1·028?

13. Two forces whose magnitudes are in the ratio of 3 to 4 act at right angles to each other; what is the magnitude of their resultant?

14. Two equal forces have a resultant also equal to either of them; at what angle are the two components acting?

15. It is required to substitute for a given vertical force two forces, one horizontal and the other inclined at an angle of 45° to the vertical. Find the ratio of the two components to the original force.

16. If a body is falling down an inclined plane, show how to compute the part of gravity which is acting upon it in the direction of motion.

17. A body is falling down an inclined plane without friction, the angle of elevation of the plane being 30°. Find the space it will pass once in the two first seconds from rest.

18. Two weights are attached to the ends of a string without weight, and are slung over a smooth pulley. Give an expression for the acceleration acting on the system if it is free to move.

19. If the weights in the preceding question be 20 and 10 gms., how many cms. will the larger weight have fallen from rest at the end of 3 secs.? *Ans.* 1471·5.

20. Illustrate the meaning of the term work by giving a list of examples of cases in which work is done *on* a body, and also a list of cases in which work is done *by* a body.

21. Discuss the principle of conservation of energy as applied (i) in the case of two bodies moving with mutual friction; (ii) in the case of impact between two bodies.

22. Show how the sun's energy is employed to grind corn, (i) by means of a wind-mill; (ii) by means of a water-mill.

23. Two balls M, M', moving in the same line with velocities V and V', impinge. Show that there will be a loss of energy during the impact unless the whole momentum exchanged be equal to $\frac{2MM'}{M+M'}(V-V')$.

24. Show that the energy stored up in a reservoir of water standing on the ground is measured by half the product of its depth into its weight.

25. Show that if an additional quantity of water is to be added to the reservoir, it will be immaterial whether it is raised up and poured in from above or forced in at the bottom.

26. Show also from the principle of conservation of energy, that if an orifice be made in the bottom of the reservoir and the water escape without friction, the velocity of the issuing stream will be that due to a fall under gravity from a height equal to the depth of the water.

27. In ques. 26 what would be the velocity if the vessel were filled with mercury whose density is 13 times that of water?

28. Prove that the work done in lifting a body up an inclined plane is equal to the work done in lifting it vertically through the height of the plane.

29. Compare the momenta of a cannon ball of 600 lbs. moving at the rate of 1000 feet per second, and that of an express train of 100 tons moving at the rate of 40 miles per hour. *Ans.* Ratio of 225 to 4928.

30. Compare also the work done in stopping the cannon ball and train in the preceding question.

Ans. Ratio of 27 to 34·7 nearly.

31. A block of wood weighing one cwt. less 4 oz. is suspended by a string, and is struck horizontally by a bullet weighing 4 oz., which sinks into the block and causes it to ascend six inches. Calculate the velocity of the bullet in feet per second. *Ans.* $1792\sqrt{2}$.

32. A bullet moving at the rate of 1000 feet per second penetrates three inches into a fixed block of wood. Calculate the velocity necessary to cause it to penetrate eight inches.

33. Compare the masses of two cylindrical bullets which proceeding from cannons of the same bore, with the same velocity, penetrate 8 and 12 inches respectively into the same block of timber.

34. A bullet weighing 240 grammes, and moving with a velocity of 300 metres per second, penetrates 4 cms. into a block of timber, the area of section being 8 sq. cms. Compare the retardation of the timber per sq. cm. with the acceleration of gravity. *Ans.* 1792 to 1 nearly.

35. Find the kinetic energy of a ring which makes a given number of revolutions per second round an axis passing through its centre and perpendicular to its plane.

Let a be the radius and m the mass of the ring: also let it make n revolutions per second.

Each particle of ring describes $2\pi an$ cm. per sec.;

$$\therefore \text{velocity of each particle} = 2\pi na,$$

$$\text{and mass of ring} = m.$$

$$\text{Hence kinetic energy} = \tfrac{1}{2}m \,.\, (2\pi na)^2$$

$$= 2\pi^2 n^2 m a^2.$$

36. Find the kinetic energy of a solid disc revolving about an axis through its centre making n revolutions per second.

The disc may be regarded as made up of a series of narrow rings. If O be the centre, and OPQ be drawn through one of the rings, the mass of the ring $= 2\pi\rho . OP . PQ$, where ρ is the mass per unit of area.

$$\begin{aligned}\text{Hence energy of ring} &= 2\pi^2 n^2 OP^2 . (2\pi\rho . OP . PQ) \\ &= 4\pi^3 \rho n^2 . OP^3 . PQ \\ &= \pi^3 \rho n^2 . 4\, OP^3 (OQ - OP) \\ &= \pi^3 \rho n^2 (OQ^4 - OP^4), \text{ see Art. 31*};\end{aligned}$$

whence adding all the successive elements,

$$\text{kinetic energy of disc} = \pi^3 \rho n^2 a^4 = \text{mass} \times \pi^2 n^2 a^2.$$

CHAPTER II.

THEORY OF POTENTIAL.

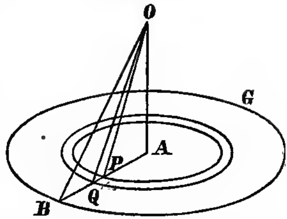
30. *Law of Inverse Squares.* It is found by experiment and observation that between every two particles of matter in the universe there exists an attraction, which depends only on the masses of the particles and on the distance between them. As the distance increases, the force of attraction diminishes according to a law called the law of inverse squares. Thus if the distance be doubled, the force is reduced in the ratio 1 to 4, or $\frac{1}{2^2}$; if the distance be trebled, the force is only one ninth or $\frac{1}{3^2}$ of its initial value, and so on. This is expressed by saying that if the masses are m and m', and the distance r, the force of attraction between them is $\frac{mm'}{r^2}$. Thus the acceleration produced by m on a particle at a distance r is $\frac{m}{r^2}$. We shall now investigate two very important cases of attraction coming under this law.

31. Prop. I. To find the attraction of a thin circular plate on a particle placed in a line perpendicular to it through its centre.

Let us suppose the plate divided into very narrow circular rings, drawn about its centre A (fig. 6), and let PQ be a type of such rings. We shall consider the attraction of each ring separately, and add them together to find that of the whole plate. Let O be the point, at which we will suppose a unit of mass (gramme) to be placed. Join OA. The resultant attraction will be by symmetry along OA. We shall

therefore resolve each force along that line, and add together the resolved parts so obtained.

Fig. 6.



Now all parts of each ring will be at the same angular distance from OA and will also be at the same distance from O. Let the radius AB cut one ring as at P and Q, where P, Q are points on its inner and outer edge respectively. Join OB, OQ, OP.

If we take an element of the ring of mass m, its attraction lies between $\frac{m}{OP^2}$ and $\frac{m}{OQ^2}$, which it would be were the mass collected at P or Q respectively. *We shall assume the attraction to be* $\frac{m}{OP.OQ}$. Again the direction of this resultant attraction will lie between OP and OQ. *We shall assume it to be towards a point R in PQ such that*

$$OR = \tfrac{1}{2}(OP + OQ).$$

Thus the attraction of the element on the gramme at O

$$= \frac{m}{OP.OQ} \cdot \cos ROA = \frac{m}{OP.OQ} \cdot \frac{OA}{OR}.$$

Adding together all the elements of the ring, the attraction of the ring

$$= \frac{\text{mass of ring}}{OP.OQ} \cdot \frac{OA}{OR}.$$

But the mass of the ring (ρ being the mass of a unit of area)

$$= \pi\rho\,(AQ^2 - AP^2)$$
$$= \pi\rho\,(OQ^2 - OP^2)$$
$$= \pi\rho\,(OQ + OP)\,(OQ - OP);$$

$\therefore$ the attraction of the ring

$$= \frac{\pi\rho\,(OQ + OP)\,(OQ - OP)}{OP.OQ} \cdot \frac{OA}{OR}$$

$$= 2\pi\rho \cdot \frac{OQ - OP}{OP.OQ} \cdot OA;$$

since $\qquad OR = \tfrac{1}{2}(OP + OQ)$

$$= 2\pi\rho \,.\, OA \,.\left(\frac{1}{OP} - \frac{1}{OQ}\right)$$

Now suppose AB to be divided up into a very large number n of such rings, which cut AB in $P_1, P_2, P_3, \ldots P_{n-1}$. To each of such rings the above formula applies, and the attraction of the whole plate which equals the sum of the attraction of all the separate rings

$$= 2\pi\rho . OA . \left[\left(\frac{1}{OA} - \frac{1}{OP_1}\right) + \left(\frac{1}{OP_1} - \frac{1}{OP_2}\right) + \left(\frac{1}{OP_2} - \frac{1}{OP_3}\right) + \ldots + \left(\frac{1}{OP_{n-1}} - \frac{1}{OB}\right)\right]$$

$$= 2\pi\rho OA \left(\frac{1}{OA} - \frac{1}{OB}\right)$$

$$= 2\pi\rho \left(1 - \frac{OA}{OB}\right)$$

$$= 2\pi\rho\,(1 - \cos\alpha),$$

where α = half the angular diameter of the plate as seen from O.

If the plate be of very large extent or the particle at O very near to it, α will become very nearly a right angle, and its cosine will be so small that it may be neglected compared with unity. Hence the attraction of any plate on a unit-mass at a distance from it, very small compared with its diameter, is always $2\pi\rho$.

In the above process two assumptions are made which are italicised.

It remains for us to show that the error in each element cannot on summation mount up to a significant term in the result.

Both assumptions consist in assigning to a term a value intermediate between the extreme values, of which the geometry showed it capable.

The error therefore in the case of any element cannot exceed the difference of these extreme values. Hence the

whole error in estimating the attraction of the ring PQ is less than

$$\frac{\text{mass of ring}}{OP^2} \cdot \frac{OA}{OP} - \frac{\text{mass of ring}}{OQ^2} \cdot \frac{OA}{OQ},$$

or $$< \pi\rho \,.\, OA \,.\, (OQ^2 - OP^2)\left(\frac{1}{OP^3} - \frac{1}{OQ^3}\right),$$

or $$< \pi\rho \,.\, OA \,\frac{(OQ^2 - OP^2)(OQ - OP)(OQ^2 + OP\,.\,OQ + OP^2)}{OP^3\,.\,OQ^3}$$

or much more

$$< \pi\rho \,.\, OA \,.\, \frac{(OQ^2 - OP^2)(OQ - OP)(OQ + OP)^2}{OP^3\,.\,OQ^3}$$

or $$< \pi\rho \,.\, OA \,.\left(\frac{1}{OQ} + \frac{1}{OP}\right)^3 (OQ - OP)^2.$$

But OP and OQ are both greater than OA.

Hence error $< \pi\rho OA \left(\dfrac{2}{OA}\right)^3 (OQ - OP)^2$.

Let now the n rings be chosen so that $OQ - OP$ is the same for each, so that $n\,(OQ - OP) = AB$.

Hence whole error $< n\pi\rho OA \left(\dfrac{2}{OA}\right)^3 (OQ - OP)^2$,

or $$< \frac{8\pi\rho}{OA^2} \,.\, AB\,(OQ - OP).$$

Now the number of the rings can be made as great as we please and therefore $OQ - OP$ can be made in all cases indefinitely small, and it is clear that the whole error committed cannot exceed a quantity itself indefinitely small, which may therefore be neglected. The proposition is now completely established.

31*. Note. As the method used in the above Article will enter largely into our future investigations it may be expedient to give here a general statement of the nature of the process.

We have generally to sum a series of the form

$$u_0 + u_1 + u_2 + \ldots + u_{n-1},$$

where all we know about the successive terms is that they are certain very small quantities each of which lies between very narrow limits, defined geometrically: while all we know about n is that it is a very large number. Our method consists in putting u_0, u_1, u_2, &c. in the form

$$u_0 = x_0 - x_1,$$
$$u_1 = x_1 - x_2,$$
$$u_2 = x_2 - x_3,$$
$$\dots\dots\dots\dots$$
$$u_{n-1} = x_{n-1} - x_n.$$

Hence on addition the sum of the series

$$= x_0 - x_n.$$

Now in transforming u_0 for instance into $x_0 - x_1$, we generally take for u_0 any value between its extreme values which can easily be decomposed into the form indicated. It is necessary to show that no appreciable error is introduced into the result. Suppose $\frac{a}{b}k$, $\frac{a'}{b'}k$ to be the extreme values of which one of the terms u is susceptible. Here $\frac{a}{b}$ and $\frac{a'}{b'}$ are fractions whose numerator and denominator differ by small quantities easily expressed as multiples of k, so that we can assume $a' = a + pk$, $b' = b + qk$; and k is itself a very small quantity. Hence the error committed in this term cannot exceed *numerically*

$$\left(\frac{a}{b} - \frac{a + pk}{b + qk}\right) k,$$

which we will for simplicity assume positive. The error therefore is certainly less than

$$\frac{aqk - bpk}{b\,(b + qk)} \cdot k,$$

or
$$< \frac{aq - bp}{bb'} \cdot k^2.$$

Let us now choose the successive terms so that k may be the same for each and $nk = K$ some finite quantity. Also on this hypothesis let $\frac{aq - bp}{bb}$ have its greatest possible

value C, suppose, which will certainly be finite as neither b nor b' vanishes. Then the whole error will certainly be less than nCk^2 or CKk.

But by making the number of terms sufficiently great k can be made indefinitely small, and hence the term CKk is also indefinitely small.

This shows that the error committed can in no case rise to importance in the final summation.

The same reasoning holds good if the value assumed for u does not lie between its extreme values, provided the greatest possible error be some finite multiple of k^2.

The following applications of this method will be commonly applied hereafter:

(i) If k represent a small angle we shall assume that its sine, circular measure, and tangent are interchangeable.

(ii) If k represent a small arc we shall assume that the chord may be substituted for the arc and *vice versa.*

(iii) If k be a small fraction we shall assume that we may substitute for it any convergent algebraical series whose first term is k, as for instance $\log(1+k)$ or $-\log(1-k)$.

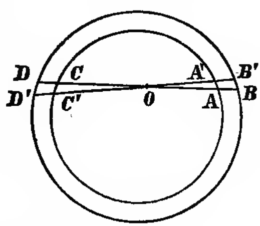
The student of Calculus will at once see that these sums are in reality definite Integrals, and the terms rejected are terms of the second order in the differential.

32. Prop. II. A spherical shell exercises no attraction on a particle placed in its interior.

Suppose a gramme to be placed at O, a point in the interior of the shell.

Then draw through O a double cone of small vertical angle. The intersection of the cone with the shell cuts off two small frusta AB', CD' from the cone. The attractions of these two small elements of the shell, on the particle at O, are exerted in opposite directions, and the resultant attraction towards A is

Fig. 7.



$$\frac{\text{mass of } AB'}{OA^2} - \frac{\text{mass of } CD'}{OC^2}.$$

But since the tangents drawn to the sphere AA', CC' are equally inclined to AOC, we may consider AB' and CD' as parallel plates of equal thickness cut from a cone, and in this case

$$\frac{\text{volume of } AB'}{\text{volume of } CD'} = \frac{OA^2}{OC^2},$$

and the volumes are proportional to the masses, since the shell is homogeneous;

$$\therefore \frac{\text{mass of } AB'}{\text{mass of } CD'} = \frac{OA^2}{OC'^2},$$

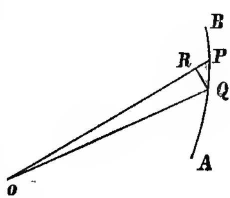
$$\therefore \frac{\text{mass of } AB'}{OA^2} = \frac{\text{mass of } CD'}{OC^2}.$$

Hence the resultant attraction of the two opposite elements on O is nil.

Now if the whole shell be cut up into similar pairs of elements, the same reasoning will hold good for each pair, and the whole attraction of the shell on any internal point vanishes.

33. Prop. III. To find the work done in carrying a gramme against the attraction of any system of particles from one point to any other point.

Fig. 8.



Let PQ be a small element of the path pursued, and let a mass of matter, m, be placed at O. Join OP, OQ, and from Q draw QR perpendicular to OP.

Then the attraction on the gramme anywhere between P and Q is represented by $\frac{m}{OP.OQ}$ for the reason given in Prop. I.

This attraction resolved along PQ

$$= \frac{m}{OP.OQ} \cos OPQ$$

$$= \frac{m}{OP.OQ} \cdot \frac{PR}{PQ}.$$

Hence the work done in carrying the gramme from Q to P is

$$\frac{m}{OP.OQ}\cdot\frac{PR}{PQ}\cdot PQ$$
$$=\frac{m\,PR}{OP.OQ}=\frac{m\,(OP-OQ)}{OP.OQ}$$
$$=m\left(\frac{1}{OQ}-\frac{1}{OP}\right).$$

If there be other particles in the system, it is clear that the acceleration in direction PQ is equal to the sum of their separate accelerations. Hence work done from Q to P

= (total acceleration along PQ) $\times$ PQ;

= sum of work done against each separately.

Hence the work done against the attraction of a system of particles m_1, m_2,...... placed at points O_1, O_2,...... may be expressed by

$$\Sigma m\left(\frac{1}{OQ}-\frac{1}{OP}\right).$$

In the same way, dividing the whole arc into similar elements, and performing the summation, we find that the work done against any attracting system in carrying a gramme from A to B

$$=\Sigma m\left(\frac{1}{OA}-\frac{1}{OB}\right)$$
$$=\Sigma m\left(\frac{1}{r}-\frac{1}{R}\right).$$

This shows us that the whole work done against any attracting system in moving a body from A to B along any path whatever is the sum of work done along the whole path (or along different paths terminating in A and B), against each element of the system taken separately.

34. If the particle move freely from B to A under the influence of the attracting system, the law of Kinetic energy must hold (see Art. 24), and we have, if v, V be the final and initial velocities,

$$\frac{1}{2}Mv^2-\frac{1}{2}MV^2=M\Sigma m\left(\frac{1}{r}-\frac{1}{R}\right),$$

where M is the mass of the particle moved, and m is, as above, the mass of one of the attracting particles. This shows us that both the work done, and the change in Kinetic energy between any two points are independent of the path pursued, depending only on the initial and final positions. If the body be carried to a great distance from the attracting mass, $\frac{1}{R}$ may be neglected, and we have for the work done in carrying a gramme from a given point to an infinite distance from that point along any path whatever, the expression

$$\Sigma \frac{m}{r},$$

where m is the mass of one of the attracting particles, and r its distance from the point.

35. Def. XI. Potential. *If a point be at distances* $\mathrm{r}, \mathrm{r}_1, \mathrm{r}_2 \ldots\ldots$ *from any given points in space, at which we will imagine masses* $\mathrm{m}, \mathrm{m}_1, \mathrm{m}_2 \ldots\ldots$ *placed, the sum*

$$\frac{\mathrm{m}}{\mathrm{r}} + \frac{\mathrm{m}_1}{\mathrm{r}_1} + \frac{\mathrm{m}_2}{\mathrm{r}_2} + \&\mathrm{c}.,$$

or, as it is written, $\Sigma \frac{\mathrm{m}}{\mathrm{r}}$, *is defined as the potential* at *that point due to that attracting system.*

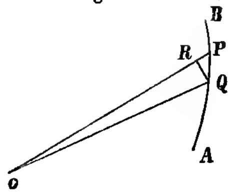
We have already shown that $\Sigma \frac{m}{r}$ measures the work done in carrying a gramme from the given point in any direction whatever to an infinite distance from the source of attraction.

The potential at a point may therefore also be defined as the work done in carrying a gramme from that point to infinity.

36. Prop. IV. If the potential of an attracting system be measured at any two points, the difference between its values is equal to the work done on a unit of mass which is moved from one point to the other.

This might be inferred from the previous proposition, but we give an independent proof.

Fig. 8*



Let A and B be the two points, and m be one particle of the system situated at O. Let PQ be a short piece of any path from A to B.

Then if V_P be the potential at P, and V_Q the potential at Q due to particle m,

$$V_Q - V_P = \frac{m}{OQ} - \frac{m}{OP} = m\frac{OP - OQ}{OQ \, . \, OP}$$

$$= \frac{mRP}{OQ \, . \, OP} = \frac{m}{OQ \; OP} \, . \, \frac{RP}{PQ} \, . \, PQ$$

$$= \frac{m}{OQ \, . \, OP} \, . \cos RPQ \, . \, PQ.$$

Now the average attraction of m on the unit of mass throughout PQ will be intermediate between $\frac{m}{OP^2}$ and $\frac{m}{OQ^2}$, and may be taken as before as the geometric mean, that is, as

$$\frac{m}{OP \; OQ};$$

$$\therefore \; \frac{m}{OP \, . \, OQ} \cos RPQ$$

is the value of the acceleration along $PQ = f$, suppose;

$$\therefore \; V_Q - V_P = f \, . \, PQ,$$

but the right-hand side of this equation equals the work done against an acceleration f in moving a unit of mass from P to Q, and this is therefore equal to the difference of potential at these points.

The same equation will be true for all the particles of the system;

$$\therefore \Sigma V_Q - \Sigma V_P = \Sigma f \, . \, PQ.$$

Or the whole change in potential, due to the system, is equal to the work done in carrying a unit of mass from Q to P. The same will hold true for all the small elements into which the space AB may be divided, and the sum of all

the differences of potentials will be the whole difference of potential between A and B.

Hence for any attracting system, the difference of potential between any two points in its neighbourhood will be the work done in carrying a gramme from one point to the other.

37. Prop. V. The acceleration of any system of particles, on a mass in any position, measured in any direction, is equal to the rate of change of potential in that direction.

We have already established in the previous proposition the formula

$$\Sigma V_Q - \Sigma V_P = \Sigma f \,.\, PQ,$$

where ΣV_P is the whole potential at P, and ΣV_Q the whole potential at Q, a point near it.

It follows of course that

$$\Sigma f = \frac{\Sigma V_Q - \Sigma V_P}{PQ}.$$

Σf is the whole acceleration in the direction PQ, and since the potential changes from ΣV_P to ΣV_Q in the space PQ, $\dfrac{\Sigma V_Q - \Sigma V_P}{PQ}$ must measure the rate of change of potential in the direction PQ.

38. Def. XII. Lines of Force. *Lines of force are lines such that the tangent at any point represents the direction of the resultant attraction of a system of particles, on a particle placed at the given point.*

Hence if a particle be free to move, it will begin to move along a line of force.

In the case of a single attracting particle, its lines of force are clearly straight lines, radiating from the particle.

39. Def. XIII. Tubes of Force. *A tubular surface bounded by lines of force is called a tube of force.*

In the case of a single particle the tubes of force will be cones of any shape originating in the particle.

40. Def. XIV. Equipotential Surfaces. *Equipotential surfaces are a system of surfaces drawn round an attract-*

ing system such that the potential at any point on one of the surfaces is always the same.

The equipotential surfaces in the case of a single particle will be a system of concentric spheres, having the given particle as their common centre.

41. Prop. VI. Lines of force are everywhere perpendicular to equipotential surfaces.

For if not, let a line of force be inclined to the equipotential surface through the point. Then the resultant force at that point has a component *along* the surface, and the magnitude of this component is the rate of change of potential in this direction (by Prop. V). But everywhere on an equipotential surface the potential is constant, and its rate of variation therefore zero, hence the resultant force can have no component along the surface, or, in other words, is perpendicular to the surface.

42. Cor. Since the resultant force is greater than any component $\dfrac{V_Q - V_P}{PQ}$ will be greatest along a line of force. Hence the line of force is the line along which the potential changes most rapidly; therefore by definition of a line of force, a body will begin to move in the direction in which potential diminishes fastest.

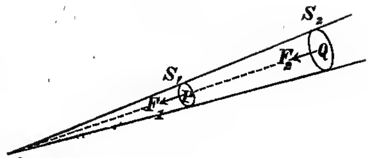
43. Prop. VII. If a cone of very small vertical angle be drawn having a particle of attracting matter at its vertex. If F be the attraction at any point within the cone computed in any direction, and S the area of the section of the cone perpendicular to the direction of F, then the product FS is constant throughout the cone.

44. (i) Let the direction of the force be along the axis of the cone, the sections are then at right angles to the cone.

Let P, Q be two points on the axis of the cone, and F_1, F_2 the attractions exerted by O on them; S_1, S_2 the sections of the cone through PQ perpendicular to the axis.

Then

Fig. 9.



$$F_1 : F_2 :: \frac{m}{OP^2} : \frac{m}{OQ^2},$$

where m is the mass of the attracting particle at O.

And since S_1, S_2 are similar figures,

$$S_1 : S_2 :: OP^2 : OQ^2,$$

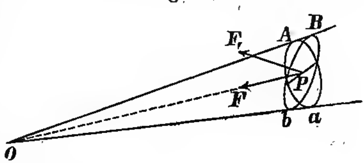
$$\therefore F_1S_1 : F_2S_2 :: m : m;$$

hence in this case $F_1S_1 = F_2S_2$.

45. (ii) Let the force be inclined to the axis of the cone at any angle θ, the section is then oblique and inclined at an angle θ to the right section.

Let F and Aa (S) be the resultant force and right section at any point P: also let F_1, S_1 be the same quantities for an oblique section Bb through the same point.

Fig. 10.



Then we may regard Aa as the orthogonal projection of Bb, and the inclination of the two sections being θ, we have

$$Bb \cos\theta = Aa,$$

$$\text{or, } S_1 \cos\theta = S.$$

Again, since F_1 is the resolved part of F in a direction inclined at an angle θ,

$$F_1 = F\cos\theta;$$

$$\therefore F_1S_1 \cos\theta = FS\cos\theta,$$

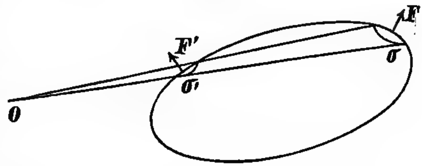
$$\text{or, } F_1S_1 = FS,$$

which proves the proposition generally.

46. Prop. VIII. If the area of a closed surface be divided into a large number of elements σ_1, σ_2, σ_3 ..., and the force of an attracting system outside it be computed over each elementary area and normal to it reckoned outwards, the sum denoted by the symbol $\Sigma F\sigma$ shall vanish.

Take one element of the attracting matter, as O, and draw from O a small cone which cuts the surface in two elementary areas σ, σ', and let F, F' be the forces normal to the surface computed outwards. Then by the preceding proposition

Fig. 11.



$$-F\sigma = +F'\sigma':$$

the sign $-$ being attached because the normal components at σ and σ' are in opposite directions with respect to O, that is, one tends *to* O and the other *from* O;

$$\therefore\ F\sigma + F'\sigma' = 0;$$

and since the whole surface can be cut up into similar pairs of elements we have over the whole surface $\Sigma F\sigma = 0$.

Again, what is true for each particle of the attracting mass outside, taken separately, is true when they are all taken together. Hence, if F_1, F_2, &c. represent the resultant force due to the whole external mass on each element of the surface, we must also have over the whole surface as before $\Sigma F\sigma = 0$.

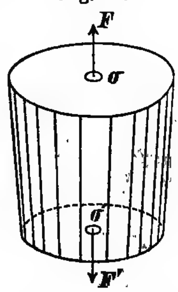
This proposition, as well as substantially the proof here given, is due to ~~Prof. Stokes~~.

47. **Prop. IX. If a tube of force, bounded as to its ends by two equipotential surfaces, have the ends divided into elements σ_1, σ_2, σ_3, &c., σ_1', σ_2', σ_3', &c., and the resultant force computed over each element $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, &c., $\mathbf{F}_1'$, $\mathbf{F}_2'$, $\mathbf{F}_3'$, &c.; then**

$$\Sigma\mathbf{F}\sigma = \Sigma\mathbf{F}'\sigma'.$$

For the tube of force so bounded is a closed surface, and we may apply to it the statement of the preceding proposition.

Fig. 12.



Now since the tube of force is bounded by lines of force, and since a force can produce no effect in a direction at right angles to itself, the component of the force perpendicular to the surface at every point on the tubular surface is zero.

Hence we have only to consider the force on the ends of the tube, that is on the equipotential surfaces, and we have

$$\Sigma F\sigma - \Sigma F'\sigma' = 0,$$

the − sign being used because the direction of the force on one surface is inwards and on the other outwards, with respect to the portion of the tube of force under consideration.

$$\therefore\ \Sigma F\sigma = \Sigma F'\sigma'.$$

If the force is uniform over the ends or equipotential surfaces then we may write the equation as

$$FS = F'S',$$

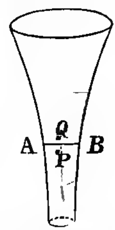
where S is the whole area of one end or cross section and S' of the other, and F, F' are the forces over each respectively.

This expresses the fact that in any portion of a tube of force the force varies inversely as the cross section of the tube, or the product of force by area of cross section is constant. This will be seen to be of great importance in electrical problems.

43. Prop. X. If a small tube of force cut through a thin plate of attracting matter perpendicular to it, the product Fσ in passing from one side to the other changes by 4πm, where m is the mass of matter included in the tube.

For let AB be the thin plate of matter included, and let P and Q be two points taken very near the plate and on opposite sides. We will denote all the attracting matter outside the plate by M, the tube of force being due jointly to the attraction of M and the plate AB. Since the forces due to M and to AB are at P and Q both perpendicular to AB, we may by the second law of motion consider their effects separately and add them together.

Fig. 13.



Let F' be the resultant attraction of M on P or Q which are indefinitely near together, the direction of F' we will suppose in the figure from Q to P. The attraction of the plate will be $2\pi\rho$ (Art. 31). At P the attraction of the plate acts *against* F', and hence the complete product $F\sigma$

$$F\sigma = (F' - 2\pi\rho)\,\sigma.$$

At Q the attraction of the plate acts *with* F', and hence the complete product $F\sigma$

$$= (F' + 2\pi\rho)\,\sigma.$$

Hence the change in the product $F\sigma$ on passing from one side of the plate to the other

$$\begin{aligned} &= (F' + 2\pi\rho)\,\sigma - (F' - 2\pi\rho)\,\sigma \\ &= 4\pi\rho\sigma \\ &= 4\pi m, \end{aligned}$$

since where $\rho =$ density per unit of area and σ the area of the plate $m = \rho\sigma$.

The same proposition holds true if the mass be not a thin plate, since we may conceive it to be made up of thin plates cut perpendicularly by tubes of force with spaces between; the above proposition is true for each plate separately, and consequently it is true for any mass cut through by a tube of force.

49. Prop. XI. As we recede from an attracting mass the equipotential surfaces tend to become a system of concentric spheres over each of which the force is uniform.

As we get to places distant from the centres of attraction the lines of attraction to different parts of the system become sensibly parallel and the ratio of the distances from the different parts of the system sensibly unity.

Hence the lines of force will be straight lines emanating from the centre of gravity of the system, and the equipotential surfaces therefore spheres having that centre of gravity for their centre. In such a case throughout a very limited space compared with the whole distance we might treat the lines of force as parallel lines, the equipotential surfaces as planes, and the force as uniform throughout that space.

We may illustrate this case by the force of gravity near the earth's surface. Over a limited area the lines of force being vertical are treated as parallel, and the equipotential surfaces as horizontal planes.

50. The difference of potential between two points is the work done in carrying a gramme from one place to the other. But by Definition 8 this is represented by the product of acceleration into height.

Hence the difference of potential of any two points whose difference of height is h centimetres is 981 h.

We may therefore say that difference of potential is difference of level, if for a moment we make our unit of length 981 centimetres.

It is clear there is no such thing as absolute level. But we find it convenient to refer all levels to that of the sea, and to call heights above the sea level positive, and below the sea level negative.

But yet we may use the terms positive and negative level or potential in respect of two places quite regardless of their absolute levels or potentials.

Thus we might say that the Mediterranean Sea has negative potential or level relatively to Mount Ararat, but positive potential or level relatively to the Dead Sea.

EXAMPLES ON CHAPTER II.

1. Show that in computing the attraction of a solid sphere on a point within its mass we may neglect all of the sphere more remote from the centre than the given point.

2. Given that the volumes of spheres are proportional to the cubes of their radii, show that the attractions exerted by a sphere on points within it are directly proportional to their distances from the centre.

3. Show that, supposing the density of the earth to be uniform and its diameter doubled, the acceleration at its surface would be double its present value.

4. If three particles of masses m_1, m_2, m_3 be placed at the angular points of a triangle, the potential at the centre of the circumscribing circle is $\frac{m_1 + m_2 + m_3}{R}$, where R is the radius of the circle.

5. If any number of particles be distributed over the surface of a sphere, the potential at the centre of the sphere is $\frac{\Sigma m}{R}$, where Σm is the sum of all the masses and R the radius of the sphere.

6. At the angular points of a triangle are placed masses equal numerically to the lengths of the opposite sides. Show that the potential at the intersection of perpendiculars is equal to $\tan A \tan B \tan C$; at the centre of the circumscribed circle it is $8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ and at the centre of the inscribed circle

$$\frac{8R}{r}\left\{\sin\frac{\pi - A}{4}\sin\frac{\pi - B}{4}\sin\frac{\pi - C}{4} - \sin\frac{\pi - 3A}{4}\sin\frac{\pi - 3B}{4}\sin\frac{\pi - 3C}{4}\right\},$$

where R and r are the radii of circumscribed and inscribed circles.

7. Calculate the potential of a circular plate on a point situated on a line through its centre perpendicular to its plane.

Ans. $2\pi\rho l(1 - \cos\alpha)$; if l be distance from edge and 2α the angle subtended by the plate.

8. Show that in a field of uniform force the lines of force are parallel straight lines and the equipotential surfaces a system of parallel planes.

9. Show that if the equipotential surfaces be a system of concentric spheres the force over each sphere is uniform.

10. Show that the attractions of all parallel plates of equal thickness cut from a ~~right~~ cone on a particle placed at the vertex are equal.

11. Show that the last proposition is true for sections taken from an oblique cone.

12. Two similar right cones of like material attract equal particles placed at their respective vertices. Prove that the attractions are proportional to the heights.

13. Find the potential of a solid right cone on a particle at its vertex.

Ans. $\pi l^2 \cos\alpha(1 - \cos\alpha)$; if l be the slant height and α the semi-vertical angle.

14. If particles be placed at the middle points of the sides of a triangle, their masses being numerically the same as the sides, show that the potential at the centre of circumscribing circle is $2 \tan A \,.\, \tan B \,.\, \tan C$.

15. If equal particles of matter be placed round an ellipse at distances such that the angle subtended between any two successive particles at the focus is constant, show that the potential at the focus is $\frac{nm}{l}$, where n is the number of particles, m the mass of each, and l the semi-latus rectum.

Note. Using the polar equation $\left(\frac{l}{r} = 1 - e \cos \theta\right)$ the proposition follows at once.

16. Find the potential of a very narrow circular annulus at its centre.

17. Find the potential of a broad circular annulus at its centre.

18. Find the potential of a sector of a circle at the centre of the circle.

19. If the equipotential surfaces be a system of confocal spheroids, show that the lines of force are systems of hyperbolas having the same foci.

CHAPTER III.

APPLICATION OF POTENTIAL TO STATICAL ELECTRICITY.

51. BEFORE proceeding to apply the properties of Potential to the investigation of Electricity, we must state briefly one or two of the experimental laws on which such application depends.

52. EXPERIMENT I. *There is no electrical force within a closed electrified conductor.*

This has been shown conclusively in numerous experiments devised by Faraday. Having tested by the proof plane, and Coulomb's balance, the inner surfaces of different conductors, of every variety of shape—spheres, cylinders, &c., with the outer surfaces either completely closed as with tinfoil, or closed only by a conducting network of wire gauze or of linen fibres, as in a butterfly net: he finally constructed a small house or room 10 or 12 feet cube, covered outside with tinfoil, and insulated on glass legs, so that the whole surface could be highly electrified by a powerful machine. Into this he carried gold-leaf electroscopes, and within it applied the most delicate tests he knew of for electrification, but he did not succeed in detecting any trace. Such was the delicacy of these tests that if there had been a ten-thousandth part of the electrification inside that there was outside he could not have failed to detect it.

Two exceptions to this law may be noted.

Exception (I). It is not true of electricity in motion.

Exception (II). An electrification may be induced inside a conducting surface, by electrified bodies insulated in cavities within it.

53. Experiment II. *When a separation of electricities takes place by friction or any other means, the amounts of vitreous and resinous electricities produced are always such that, on being re-united, they exactly neutralize each other.*

This is shown clearly in any form of electrical machine in which the opposite poles are connected. For unless it were true one pole would, on working the machine, still acquire a charge of electricity.

Def. Complementary Distributions. *The two amounts, which are produced when the electricities of a neutral body are separated, are said to be equal and of opposite sign, and we shall speak of them as complementary distributions.*

54. Experiment III. *The amount of opposite electricity induced on surrounding conductors by any electrified body is equal to the body's own charge.*

This is experimentally proved by Faraday's Ice-pail experiment. An electrified sphere is introduced into a hollow closed conductor, and the electricity induced on the *inside* by the charged body *before contact* of the sphere with the interior, is found *on contact* just to neutralize the body's charge. The complementary distribution on the outside, which is equal to the disguised or induced charge, must therefore be equal to the original charge of the body.

55. Experiment IV. *If two bodies be electrified and placed at a constant distance, great compared with their dimensions, from each other; they exert on each other a force proportional to the products of the amounts of electricity they contain. This force is attractive if their electrification be opposite, repulsive if similar.*

We can measure the repulsion of two charged bodies by Coulomb's torsion balance, in which the moment of the repulsive or attractive force is equal to the torsion of the wire required to keep the bodies at a fixed distance.

The charges can be varied in the following manner: Provide a ball of the same size as the carrier and indicator balls of the torsion balance, insulated by a silk thread or gum-lac stem, which we shall call the discharging ball.

Now, having a common frictional machine fitted with a Henley's electrometer, if we apply the carrier or discharging

ball to a certain part of the prime conductor, when the electrometer is at a fixed reading, we carry away a certain amount of electricity, which may be taken as a provisional unit.

Having charged by this means the carrier ball, it is placed in the balance; its charge is immediately divided equally with the indicator ball, and we can observe the torsion of the wire which keeps the two balls at any proposed distance apart: we have in this way a measure of the repulsive force between two quantities *each* $\frac{1}{2}$. We now remove the carrier ball, and divide its charge with the discharging ball, by which means the charge of each is reduced to $\frac{1}{4}$. The carrier ball is replaced in the balance, and the repulsion at the same distance observed.

The discharging ball is now discharged, the carrier ball removed and touched against the discharging ball, again replaced, and the repulsion at the same distance again observed.

By continuing this process we can observe the repulsion at fixed distances between quantities whose ratios are respectively 1 to 1; 1 to $\frac{1}{2}$; 1 to $\frac{1}{4}$; 1 to $\frac{1}{8}$; and so on. By this means it is found that, making certain allowances for loss of charge, the repulsion at a constant distance closely approximates to the law above given, aud on increasing the distance, the law is found more and more nearly true.

By fixing a vertical wire in the balance to prevent the indicator ball from flying to the carrier ball, and first charging the carrier ball with negative electricity, the same law can be established for the attraction of oppositely electrified bodies.

56. Experiment V. *If constant charges of electricity be condensed in two points, and the distance between them varied, the force of attraction or repulsion is found to vary inversely as the square of the distance.*

This is shown by Coulomb's torsion balance also, by varying the distance of the conductors instead of the charges. It might also be inferred from the following considerations: Let a hollow spherical shell be charged with electricity. From its symmetry of shape it is clear that the distribution of electricity over it will be uniform, and the amount on any element of its surface therefore proportional to its area.

Now, referring to Art. 32, we see that if we wish to find the electrical force at a point O within an electrified sphere we divide the surface up into opposite pairs of elements AB', CD', and then assuming the law of inverse squares, prove that there is no force at that point. But if we suppose the law of the force unknown and call it the inverse nth power of the distance, the attraction exerted by the pair of opposite elements on O (see fig. 7) will be (towards A)

$$\frac{\text{mass } AB'}{OA^n} - \frac{\text{mass } CD'}{OC^n}.$$

But $$\frac{\text{mass } AB'}{OA^2} = \frac{\text{mass } CD'}{OC^2} = k \text{ suppose.}$$

Hence the attraction on O *towards* A

$$= k\left(\frac{1}{OA^{n-2}} - \frac{1}{OC^{n-2}}\right).$$

If $n > 2$ and $OA < OC$ this result will be positive, or there will be an attraction towards the nearer side of the sphere. If $n < 2$ or negative the result is negative, showing that there will be a resultant attraction towards the more distant side of the sphere.

The whole attraction on the internal point can therefore only vanish when n exactly equals 2, i.e. for the law of the inverse square. The methods of detecting electrification are so much more delicate than any measurement by the torsion balance, that this constitutes the most reliable proof of the law, since we know as a fact that there is no electrical force anywhere within a closed electrified sphere.

57. To make our unit quantity of electricity symmetrical with our other units we adopt the following definition of electrical quantity.

58. Def. Unit Quantity of Electricity. *The unit of electricity is such a quantity, that if condensed in a point it shall exert a unit of force or one dyne on another similar unit placed at a distance of one centimetre from it.*

59. Def. Electrical Density. *Electrical density is a term used to denote the quantity of electricity on a body per square centimetre.*

Thus if a surface of area s be electrified uniformly with a charge q, and if ρ be the density at any point,

$$\rho = \frac{q}{s}$$

$$\text{or,} \quad q = \rho s.$$

If the surface be not uniformly electrified we define density as the quantity which would be on a unit of area supposing the density uniform and of the same value as at the point under consideration.

60. We have in these definitions carefully avoided any theory as to the nature of electricity.

On a fluid hypothesis the quantity measures the whole amount of electric fluid, and the density the depth of the fluid layer at any point.

61. It is usual to refer to the electricity which appears on the plate and prime conductor of an electrical machine as *positive*, while that which appears at the same time on the rubber or negative conductor is called negative. The propriety of these terms appears if we remember (Exp. 2) that the amounts developed always neutralize each other. This is conveniently expressed algebraically by saying that if quantities q and q' of electricity are developed from a neutral body by friction or otherwise $q + q' = 0$ always. Further, we have shown that if two similar electrified particles containing quantities q and q' of electricity (both + or both −) be placed at a distance r from each other, there is between them a repulsive force measured by $\frac{qq'}{r^2}$, while if the quantities q and q' be one positive and the other negative, there is an attractive force measured by $\frac{qq'}{r^2}$. Generally we may say that between any two quantities q and q' there is a repulsive force, remembering that when the product qq' is negative the repulsion becomes negative, and negative repulsion is the same as attraction.

62. Having established the fundamental proposition that between two quantities q and q' of electricity condensed in points at a distance r from each other there is a force measured

by $\frac{qq'}{r^2}$ which is repulsive if this product be *positive*, and attractive if it be *negative;* we can apply all our propositions on Potential of attracting matter to Potential of an electrical distribution.

We shall only have to substitute in our original definitions the *unit quantity of electricity condensed in a point* for the *gramme.*

Our Definition XI. Art. 35 will then stand thus.

Def. Electrical Potential. *If there be a number of points holding charges* q, q′, q″, *&c., and if there be another point at distances* r, r′, r″, *&c., from them, then the sum* $\frac{q}{r}+\frac{q'}{r'}+\frac{q''}{r''}+$ *&c.* $\left(=\Sigma\frac{q}{r}\right)$ *is defined as the potential of the given electrical distribution* at *the given point. And it is clear that as before* $\Sigma\frac{q}{r}$ *represents the work done on or by a unit of positive electricity condensed in a point when moved from the given point to an infinite distance.*

Now, when considering the attraction of masses due to gravitation, we have always to *expend or do* work in order to take a gramme to an infinite distance from any point. But from the dual nature of electricity, that is from the fact that it manifests itself in two modes, it is immediately obvious that we have here *two* cases to be dealt with.

(I). It may require work to be done to *take* the unit *from* any point to an infinite distance: or—

(II). It may require work to be done to *bring* it up from infinity *to* that point.

Assuming the testing unit always charged with positive electricity, it is obvious that for a point at which $\Sigma\frac{q}{r}$ is negative, work has to be done to take the unit from the point to infinity, and for a point at which $\Sigma\frac{q}{r}$ is positive, work has to be done to bring the unit up from infinity to the given point.

We shall hereafter call the unit of positive electricity condensed in a point a plus unit (written '+ unit').

63. Premising these extended definitions, we proceed to deduce some important results.

Prop. I. The potential over the surface and within the mass of an electrified conductor is constant.

This follows from our first experiment: for since there is no electrical force within the charged conductor there can be no change of potential, or, in other words, the potential is constant.

COR. 1. The surfaces of all electrified conductors are equipotential surfaces, and lines of force cut them at right angles.

This is equally true, whether the distribution be free or induced: if free, the distribution is such that it exerts no attraction on internal particles, and if induced, it neutralizes the force of attraction of all external electricity on internal electrified particles.

COR. 2. The law of density on a freely electrified conductor is the same as the law of thickness of a film of matter, which exerts no attraction on an internal point.

COR. 3. Whenever a difference of potential exists between two bodies, which are connected by a conductor, after some time, short or long, equality of potential is established. This is *said* to take place by a flow of positive electricity from the place of higher to that of lower potential. Its special investigation we defer till we consider electricity in motion.

64. DEF. ZERO POTENTIAL. *The absolute zero of potential exists at a place removed to an infinite distance from all electricity.*

In practice, this idea can never be realized, as all uninsulated bodies are really parts of the earth, and therefore at the same potential as the earth. Moreover, since the earth is a large conductor, and no electrical separation we can effect will sensibly alter its potential, and all electrical phenomena we observe are ultimately differences of potential between certain insulated bodies and the earth, we habitually speak of the earth as our standard of potential, and of its potential as zero potential, reserving the term absolute zero for potential at infinity.

DEF. NEGATIVE AND POSITIVE POTENTIAL. *When a place* A *is at a higher potential than a place* B, A *is said to have a potential positive to* B, *while* B *has a potential negative to* A: *the words higher and lower being taken in their algebraical sense.*

This must be clearly distinguished from positive and negative electricity. A positively electrified body is one which when removed from all other electrified bodies is positive relatively to the earth, so that on the establishment of conduction electricity flows from the body to the earth: a negatively electrified body is one whose potential is negative to the earth, the flow of electricity taking place under the same circumstances from the earth to the body.

Referring to our former illustration of level, if we have two cisterns of water at different levels, water will always flow (when a channel is opened) from the relatively higher to the relatively lower, quite independently of their absolute levels, which may both be positive, as when both are above the sea level, or one positive and one negative, as when one is above and the other below the sea level, or both may be negative, as when both are below the sea level.

Another illustration may be taken from heat, when potential corresponds to temperature. When two bodies of different temperature are brought near to each other, an interchange of heat takes place, which may be expressed as a flow of heat from the hotter to the colder body. The nominal zero of temperature is again purely conventional and has nothing to do with the absolute zero, the flow of heat merely taking place from the hotter to the colder, irrespective of absolute temperature, whether the temperatures be both positive, one positive and one negative, or both negative.

65. Prop. II. The electrical force just outside an electrified conductor at a point whose density is ρ is $4\pi\rho$.

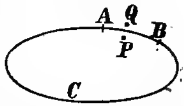
The measure of electrical force at a point is clearly (cf. Def. 4) the force on a plus-unit placed at the point. If we take a small tube of force originating in the surface of the conductor, whose area of section is σ, and the force over which is F, then $F\sigma$ is constant throughout the tube,

and on passing through the electrified surface changes by $4\pi\rho\sigma$. But within the conductor F vanishes, and hence just outside the conductor $F\sigma = 4\pi\rho\sigma$ or $F = 4\pi\rho$.

This might also be proved by considering the force within and without, near a small element of the surface, taken near the point under consideration.

Suppose AB such an element, and let P, Q be two points, one just inside, and the other just outside the surface. We may then consider the attraction of AB, and the rest of the conductor ACB, separately from each other.

Fig. 14.



It is clear that AB exerts at P a force equal and opposite to that of ACB, and since AB may be treated as a flat plate, uniformly electrified, its force at P is $2\pi\rho$ inwards.

Again, the force exerted by the part ACB at Q is the same as that which it exerts on P, and the force of AB at Q is similarly $2\pi\rho$ outwards.

Hence, the total acceleration on Q is $2\pi\rho$ due to AB, and also $2\pi\rho$ due to ACB, both outwards, or the whole force just outside the conductor is $4\pi\rho$.

The force exerted on a positive unit at Q will clearly be $4\pi\rho$ *outwards*, and on m units at Q will be $4\pi\rho m$.

Cor. The force which an electrified conductor exerts on any portion of its electrification is normal, and at the rate of $2\pi\rho^2$ per unit of area. For considering the element AB, the force due to ACB at any point on AB is $2\pi\rho$, and the quantity of electricity is $\rho\sigma$, if σ be the area of the element AB. Hence the whole force on AB is $2\pi\rho^2\sigma$. Hence the force exerted on AB is at the rate of $2\pi\rho^2$ per unit area.

66. Prop. III. If a tube of force cut through two oppositely electrified surfaces the quantities of electricity on its two ends are equal and of opposite sign.

For supposing F, σ to represent the force and area of section at one surface, and F', σ' the force *measured in same direction* and section of tube at the other surface,

$$F\sigma = F'\sigma'.$$

But $F = 4\pi\rho$, if ρ be the density on one surface,

$F' = -4\pi\rho'$, if ρ' be the density on the other surface;

$$\therefore\ 4\pi\rho\sigma = -4\pi\rho'\sigma';$$

$$\therefore\ \rho\sigma = -\rho'\sigma',$$

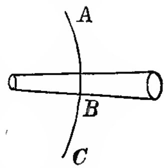
$$\text{or } q = -q',$$

where q, q' are the quantities of electricity on the two surfaces respectively.

67. **Prop. IV. It is impossible to pass from a region of constant into a region of varying potential without passing through an electrified surface.**

For let ABC be the boundary of a region of constant potential, it must therefore be an equipotential surface.

Fig. 15.

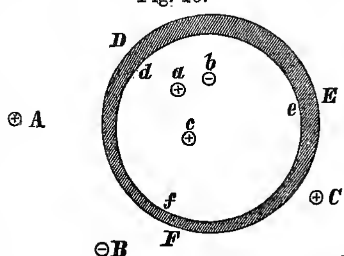


Draw any tube of force cutting it at right angles, then since $F\sigma$ is constant throughout the tube, and vanishes on one side of ABC—that of constant potential—it must vanish also on the other side of ABC, supposing it to be unelectrified. In other words the potential on the other side of ABC must also be constant, which is contrary to the supposition that potential varies.

68. **Prop. V. If a closed conducting surface be connected with the earth, there may coexist two systems of electrified bodies, one inside and one outside, entirely independent of each other.**

Suppose in the figure we have a hollow conducting body, and let there be within its inner surface def one system of electrified bodies a, b, c, and also without its outer surface DEF another system A, B, C.

Fig. 16.



The system A, B, C will induce on DEF a distribution, which will produce a constant zero potential throughout the mass of the conductor, since it is connected with the earth.

Again, the system *a, b, c* will produce no change of potential in space exterior to the surface *def* of the conducting mass, since the conductor and all external space is as far as *a, b, c* is concerned at zero potential: there will therefore only be a distribution of electricity on the surface *def* which exactly neutralizes the action of *a, b, c* on all external points.

Hence, we have two electrical systems, *a, b, c,* with its induced charge on *def* within the conductor, and *A, B, C,* with its induced charge on *DEF* outside the conductor, quite independent of each other: all actions within the conductor taking place as if *A, B, C* did not exist, and those outside as if *a, b, c* did not exist.

COR. 1. It follows from this that the amount of the charge induced on surrounding conductors by an electrified system is equal in amount and opposite in sign to the charge of the system, a result we have already indicated as established experimentally.

For if we consider a system of tubes of force originating in *a, b, c* and proceeding to the surface *def*, the amount of electricity on the opposite ends must be equal. And these tubes of force include the whole surface *def*.

Again, if tubes of force proceed from one part of the system *a, b, c* to another, they must include on their opposite ends amounts of electricity equal and opposite in sign, which consequently cancel each other in the final summation. Hence the whole amount of electricity on *def* must be equal and opposite to the whole amount of electricity in *a, b, c*.

√ COR. 2. If any electrical apparatus be placed inside a metal enclosure connected with the ground, the apparatus is entirely screened from electrical actions taking place outside it. This is of the greatest use in practical electricity, where all delicate instruments are protected by a screen of wire gauze.

69. Prop. VI. There cannot be two different laws of distribution of free electricity on a given conductor.

If possible, let there be two such laws, then they must both produce a constant potential within the conductor.

Hence, if distributions according to the two laws be superimposed on each other, the combined distribution will produce a constant potential within the conductor. Let now equal amounts of positive and negative electricity be spread over the surface according respectively to the two laws of distribution referred to. At parts they neutralize each other, and at parts there is an excess of positive electricity, in others an excess of negative electricity. Hence a free distribution, partly positive and partly negative, produces a constant potential within the conductor, a result obviously absurd.

Def. Capacity. *The quantity of electricity which will bring a conductor from zero to unit potential, is defined to be the capacity of the conductor.*

It is clear that the capacity of a conductor depends not only on the conductor itself, but on all surrounding electrified and unelectrified bodies.

70. Prop. VII. If C be the capacity of a conductor removed from all other conductors which is raised from zero to potential V by a charge Q of electricity, Q = CV.

For since the electrification of the conductor can be only according to one law it is clear that each increment in charge is spread over the conductor according to the same law, and the density at each point is altered in the same ratio. Hence the sum $\Sigma \frac{q}{r}$ or the potential will be altered in the same ratio, or the change in V is always proportional to the change in Q. But when $V = 1$, $Q = C$. Hence $Q = CV$ always.

Cor. 1. It follows that if a body be electrified with a distribution partly free and partly inductive, we may consider the effects of each separately on the potential, and simply add up the results due to each separately.

Cor. 2. It also follows that if a conductor be in a region at potential V_0, and be brought up to potential V, the quantity of the charge is $C(V - V_0)$, since the potential without any free charge is V_0. The potential of a body so electrified, examined by an electrometer entirely in the region at V_0, will be $V - V_0$, but examined by an electrometer with one pole to earth it will be V.

71. Prop. VIII. If a distribution of electricity over a closed surface produce a force at every point of the surface perpendicular to it, this distribution will produce a constant potential at every point within the surface.

Since the resultant force at any point has no component along the surface, the rate of charge of potential along the surface vanishes, and the surface is a surface of constant potential.

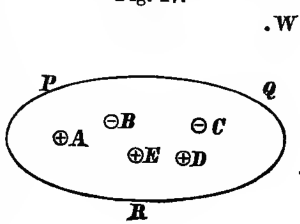
If the potential at every point within the surface be not the same, draw within it a system of equipotential surfaces and tubes of force. The equipotential surfaces can in no case cut the surface of the conductor, since the lines of force are everywhere perpendicular to it. Hence, as we proceed inwards, the successive surfaces must constantly diminish in area, and at last vanish. Conceive now any tube of force proceeding from the surface inwards. Throughout it $F\sigma$ is constant, and at some point within the surface σ must vanish, and at this point F must be infinite. But F can only be infinite at a point indefinitely near to another point, having a finite quantity of electricity, and by supposition such a point does not exist within the surface.

Hence we see that the distribution on the conductor is a possible one, and Prop. VI. shews that it is the only one.

72. Prop. IX. If an equipotential surface belonging to any electrical system be drawn, and a distribution of electricity be made over that surface such that the density at each point is $\frac{\mathbf{F}}{4\pi}$, where F is the resultant force of the system at that point, then this electrification will be in equilibrium and will produce on all external electrified particles the same force as the given electrical system.

For let ABC... be a system of electrified particles, and let PQR be an equipotential surface enclosing the system.

Fig. 17.



Replace for a moment PQR by a conducting surface communicating with the earth. This will screen an external particle W from the attraction of ABC, Prop. V. Cor. 2. This can only happen through a distribution of electricity

on PQR, which produces on W a force exactly equal and opposite to that produced by ABC....

Let now W be close to the surface, say at P. The resultant force F of ABC will here be perpendicular to the surface. Hence the resultant force due to the induced charge will also be perpendicular to the surface and will equal $-F$, and this will be true for each point on the surface: and therefore by the last proposition this induced distribution is according to the same law as a free distribution. But for a free distribution of density ρ the force just outside is $4\pi\rho$, hence

$$-F = 4\pi\rho, \text{ or } \rho = -\frac{F}{4\pi}.$$

This gives us the density of the induced distribution.

If we now distribute electricity over PQR, whose density at each point is $+\frac{F}{4\pi}$, we clearly get a distribution which produces on W a force the same in amount and direction as the original distribution ABC, and this distribution is a free distribution.

Hence we may remove the original system ABC... and replace it as far as actions outside are concerned by the equipotential surface electrified, so that its electrical density at each point is $\frac{F}{4\pi}$.

The whole amount of this electrification by Prop. V. Cor. 1 is equal to the whole amount of electricity in ABC..., and we may consequently replace the system ABC... by the conducting surface PQR freely electrified with the quantity of electricity in the whole system ABC. If the equipotential surface do not enclose the whole system but pass between two different parts the same reasoning applies if we distribute over the surface electricity equal in amount to that on the part of the system enclosed by the equipotential surface in question.

Cor. It follows that if we have any non-conducting mass and any system of electrified conductors distributed within it the resultant force on any external electricity can be represented by a distribution of electricity, partly positive and partly

negative, on the bounding surface of the non-conductor. For if we for a moment conceive the bounding surface conducting and connected with the earth a charge will be induced which screens the electrified bodies inside from all external action. If this charge be reversed in respect of positive and negative and spread over the surface of the non-conductor, this distribution satisfies the condition of the problem.

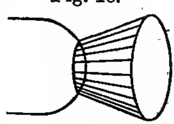
73. Prop. X. To determine the law of density over a freely electrified surface.

We have already indicated (Prop. I. Cor. 2) the means by which this can be done, but in almost every particular case the analysis baffles us.

The only method practically useful is indirect and depends on the property proved in the last Article. We draw a number of equipotential surfaces for systems of particles with different relative amounts of electricity. From such surfaces we select one which most nearly corresponds to the conductor in question. Calculating the force at each point on the surface and dividing by 4π, we get the law of density at each point of the free electrification.

We can however lay down one general rule that the electrical density of a free electrification is always greatest at places of greatest curvature. For it appears from the figure that if we take a tube of force, starting from a small area, the greater the curvature in the neighbourhood the more rapidly will the area of the tube increase. But since throughout a tube of force the product $F\sigma$ is constant (Chap. II. Prop. VII.), we see that if σ increases rapidly either F is very large near the surface or diminishes rapidly as we recede from it. Now we know the latter not to be the case, since as we recede the force tends to become equal at equal distances, (Chap. II. Prop. XI.). Hence we conclude that near a place of great curvature F is very great, and since $F=4\pi\rho$ just outside the surface it follows that ρ, the density, is also very great.

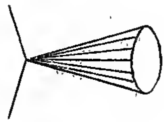
Fig. 18.



Similarly at a point or an edge it follows that the density will be theoretically infinite.

In this case the tube of force starts from a zero base, and that $F\sigma$ may be constant F must be infinite at the point or edge, or else $F\sigma$ must vanish at all distances. As a matter of fact there is no such thing in nature as a point or an edge, the parts we call such being in reality rounded off. Even if a point did exist we know that the density could not be infinite, since the air would, under high tension, cease to be an insulator, and would conduct away the electricity. This does, in fact, explain the glow always seen in the dark at sharp points when electrified.

Fig. 19.



74. We have hitherto referred to electrical actions as taking place in air, and assuming that the effects might be represented by action at a distance have made no reference to the dielectric across which these actions take place. This was the universally accepted view on the subject till Faraday by a series of experiments established the theoretical result that all actions apparently at a distance are the outcome of actions taking place in the intervening dielectric, and also that the nature of the dielectric influenced the amount of these actions.

To explain Faraday's Theory of Inductive Action we must conceive the air or other dielectric to consist of a number of conducting molecules, separated from each other by layers of insulating material. We may perhaps represent the medium as consisting of a number of small metallic shot bedded in and kept apart from each other by shellac. If now we conceive a positively electrified conductor surrounded by such a medium, the effect of the electrification is to separate the electricities in the layer next the body, each shot acquiring a positive and a negative pole, the negative pole being directed towards the conductor. This layer of shot produces an exactly similar electrical separation in the layer next to it, and so on through the whole dielectric, the poles of the consecutive molecules always being along the lines of force. The degree of electrical separation in each molecule depends on the amount of the original electrification, to which the whole amount separated over any equipotential surface is equal.

It is easily shown that the amount of electrical separation across any equipotential surface bounded by a given tube of

force is measured by $\pm \frac{1}{4\pi} F\sigma$. For if the surface become conducting the amount of electricity on the two ends of the tube is equal and opposite. If ρ be the density of the electricity on the base of the tube $F = 4\pi\rho$ and therefore $F\sigma = 4\pi\rho\sigma$, and since $\rho\sigma$ is the quantity of electricity on the base of the tube $F\sigma = 4\pi Q$ or $Q = \frac{1}{4\pi} F\sigma$. Hence the quantity separated across the equipotential surface is measured by $\pm \frac{1}{4\pi} F\sigma$, since if $+\frac{1}{4\pi} F\sigma$ is separated outwards $-\frac{1}{4\pi} F\sigma$ will be separated inwards. Also since $F\sigma$ is constant throughout the tube the same quantity is separated across any equipotential surface.

We may express this by saying that the quantity separated per unit of area across any equipotential surface is $\pm \frac{F}{4\pi}$.

The medium is by this means put in a state of strain, the lines of strain being the lines of force. The medium when strained tends to return to the normal state by a discharge of electricity from molecule to molecule, and the greater or less facility with which this is effected constitutes better or worse conduction. A good conductor cannot withstand a very small strain, while a good insulator only yields to a very violent strain. All bodies in nature fall between the limits of a perfect conductor and a perfect insulator.

We proceed to consider the effect of changing the dielectric on the electrical actions of a system.

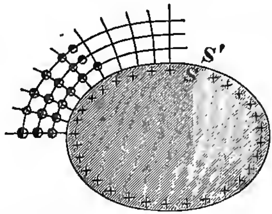
75. Prop. XI. In any given system charged with a given quantity of electricity, the effect of changing the dielectric is to alter the potential of all bodies in the system in a certain ratio.

Take the simplest case, that of a conductor immersed in a medium and freely electrified.

Supposing the medium to extend to an indefinite distance round the conductor, the whole effect of the system of electrified molecules composing the medium may be repre-

sented by a distribution of electricity over the inner surface of the dielectric (Prop. IX. Cor.).

Fig. 20.



Again, assuming the dielectric to be electrically homogeneous (isotropic), or to have no electric polarity, we see that a change in the dielectric cannot produce any change in the form of the system of equipotential surfaces.

The only effect therefore of changing the dielectric is to alter, in a certain ratio, the *effective* electrification at each point. Thus if ρ be the density at any point on an electrified conductor, conceived apart from any medium, the effective density on that point, when immersed in one medium A, becomes $\frac{\rho}{K}$, and when in another medium B, it becomes $\frac{\rho}{K'}$, where K and K' are constants depending only on the medium.

Again, since the potential at any point is $\Sigma \frac{q}{r}$, which is the same as $\Sigma \frac{\rho\sigma}{r}$, where ρ is the density on a small area σ, it is clear that the potential V when in medium A becomes $\frac{V}{K}$ and when in medium B, $\frac{V}{K'}$.

This proves the proposition for a conductor freely electrified in space, and the same method of proof can clearly be extended to any system of electrified bodies whatever, since in that case the whole system of equipotential surfaces must remain the same and that can only occur when the electrification is everywhere altered in the same ratio.

76. If we have the same conductor immersed first in a medium A and then in a medium B and brought from zero up to the same potential V, the quantities of electricity are in the ratio K to K'. For in the medium A the effective

quantity is $\frac{Q}{K}$ and therefore $\frac{Q}{K} = CV$ and $Q = CKV$. Similarly in the medium B, $Q' = CK'V$. Hence as $Q : Q' :: K : K'$.

The ratio K to K' for the two media A and B is called the ratio of their *specific inductive capacities.* Since we know nothing of the behaviour of a conductor removed from any medium we can only compare different media. Our standard of reference is air, and its specific induction is taken as a unit and the capacities of all other media compared with it.

Faraday has shewn experimentally that the specific inductive capacity of all gases whatever at all temperatures and pressures is the same, and it is this circumstance, combined with its excellence as an insulator, which makes air so convenient as a standard. He also found that for all the solid and liquid dielectrics he experimented on the specific inductive capacity was greater than for air.

DEF. SPECIFIC INDUCTIVE CAPACITY *of any dielectric is the ratio of the charge on a conductor immersed in it to the charge on the same conductor raised to the same potential in air.*

In our future investigations we shall, unless the contrary is stated, assume all actions to take place in air, our formulæ then being identical with those proved for action at a distance, and in any case where the dielectric is different from air we shall simply have to multiply the capacity of each conductor by the specific inductive capacity of the dielectric in question.

77. Prop. XII. To calculate the energy exerted in charging any conductor.

By definition the potential is the work done in bringing a + unit of electricity from zero up to the given potential, and if Q units of electricity be brought up from potential zero to potential V, the energy exerted is QV. This however is only true on the supposition that the whole amount of electricity at potential V is so large that the addition of the quantity Q does not sensibly raise the potential. If, however, Q represent the whole charge we should infer that the energy would be $\frac{1}{2}QV$, since at the beginning the potential is at zero at the end at V, and consequently the average potential is $\frac{1}{2}V$ and the whole energy exerted $\frac{1}{2}QV$.

We may shew the same result by the graphical method representing quantities by abscissæ and potentials by ordinates. Since the rise in potential is in a constant proportion to rise in quantity the extremities of the ordinates are on a straight line. If we now suppose the charge made by successive small quantities and construct the corresponding parallelograms, it is clear that the area of each parallelogram represents the amount of energy expended in raising the quantity represented by its base to the potential represented by its height. Hence the whole energy is represented by the area of the triangle or $\frac{1}{2}QV$.

We give still another proof of an algebraical kind of this very important proposition.

Let the whole charge Q be communicated to a conductor of capacity C by n different charges each equal in amount to q, so that $Q = nq$.

The potential of the first charge $q = \dfrac{q}{C}$ and energy $= \dfrac{q^2}{C}$,

........................second ...... $= \dfrac{2q}{C}$ .... $= \dfrac{2q^2}{C}$,

........................third .... $= \dfrac{3q}{C}$ .... $= \dfrac{3q^2}{C}$.

and so on.

.... n^{th} .... $= \dfrac{nq}{C}$ .... $= \dfrac{nq^2}{C}$.

Hence the whole energy exerted in charging the conductor

$$= \frac{q^2}{C} + \frac{2q^2}{C} + \frac{3q^2}{C} + \dots + \frac{nq^2}{C}$$

$$= \frac{q^2}{C} \cdot \frac{n(n+1)}{2} = \frac{(nq)^2}{C} \cdot \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}\frac{Q^2}{C}\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{2}QV\left(1 + \frac{1}{n}\right).$$

Now if the successive charges be made sufficiently small, and the number of them sufficiently great, $\dfrac{1}{n}$ may be neglected, and we get as before for the whole energy expended in charging the conductor $\frac{1}{2}QV$.

The principle of the conservation of energy shows us that the energy which runs down in the discharge is equal to the energy which is exerted in the charge, or we may prove it independently by assuming the discharge to take place by a series of n discharges of quantity q.

The sum of the energy which runs down in the successive discharges will be

$$\frac{Qq}{C} + \frac{(Q-q)}{C}q + \frac{Q-2q}{C}q + \dots + \frac{Q-(n-1)}{C}q$$

$$= n\frac{Qq}{C} - \{1+2+3+\dots+(n-1)\}\frac{q^2}{C}$$

$$= \frac{Q^2}{C} - \frac{n(n-1)}{2} \cdot \frac{q^2}{C}$$

$$= \frac{Q^2}{C} - \frac{Q^2}{2C}\left(1-\frac{1}{n}\right)$$

$$= \frac{Q^2}{2C}\left(1+\frac{1}{n}\right).$$

As before, if the number of successive discharges be sufficiently great the whole energy will be

$$\frac{Q^2}{2C} = \tfrac{1}{2}QV.$$

78. If we have any system of conductors charged with given quantities of electricity, the energy expended in charging the whole system is the sum of the energies exerted in charging the separate conductors.

It might at first sight appear that the *order* in which the different conductors of a system are charged would affect the energy, since each charge alters not only the potential of the body in question but inductively of all other bodies in the system. The conservation of energy shows however that the order of charge or discharge must be, on the whole, immaterial, as otherwise by continually charging a system in one order and discharging it in a different order there would be a gain of energy. The same principle shows us that if we charge a system of conductors, insulate them and move them about in any way relatively to each other, the whole work done against

electrical forces is the excess of the energy after the movements have taken place over the energy of the system when first electrified. We shall illustrate the use of this principle hereafter.

It may be worth noticing that the energy of charge or discharge is the same whether a body be positively or negatively electrified in free space, since for a positive electrification Q and V are both +, and for a negative electrification both −. If however Q be negative and V positive, or vice versâ, the energy of the body is apparently negative. Such an electrification can only exist when a negatively electrified body is in a region of positive potential, and the energy, kinetic or potential, acquired or expended in moving the body up into this position must be equal to the change in electrical energy of the system. As the negatively electrified body is brought up the potential of the rest of the system is lowered and its potential is raised. If the change in potential of the system just balances the energy acquired by the body it is left at zero potential. If however the energy acquired is too great the energy of the body's free electrification must be subtractive, if too little, additive.

These three cases clearly correspond to the cases in which the potential is zero, + and − respectively.

CHAPTER IV.

PROBLEMS IN STATICAL ELECTRICITY.

79. We have in the preceding chapter given demonstrations of the most important theorems on which the science of Statical Electricity rests, and we now append a series of problems, many of which are of the greatest importance to the practical electrician, while others are introduced with a view of suggesting to the student methods by which other similar problems may be successfully attacked.

80. Prop. I. To find the potential at any point within a sphere freely electrified with a known quantity of electricity.

Let R be the radius of the sphere, and Q the quantity of electricity. Since the electrification is free, the potential is constant throughout the sphere, and has the same value as at the centre.

But all parts of the electrification are at the same distance, R, from the centre, and the potential $\Sigma \frac{q}{r}$ becomes $\frac{\Sigma q}{R} = \frac{Q}{R}$.

Hence if V be the potential at any point within the sphere $V = \frac{Q}{R}$.

To find the capacity of the sphere, we have only to remember that if $V = 1$, $Q = C$;

$$\therefore \; 1 = \frac{C}{R} \text{ or } C = R.$$

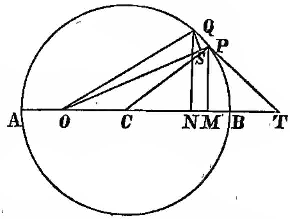
Hence the capacity is numerically equal to the radius.

We might therefore define unit capacity as the capacity of a sphere whose radius is one centimetre electrified freely to unit potential.

81. We now give a direct proof of the above proposition. Let APB be a spherical shell freely electrified with a charge whose density at any point is ρ.

Let O be any point inside the sphere, and AOB the diameter. We may conceive the sphere as generated by the revolution of a circle APB round the diameter AB.

Fig. 21.



If we take two points PQ very near each other on the circle, and draw perpendiculars PM, QN to the diameter, it is clear that PQ by its revolution traces out an annulus, whose radius is PM, and breadth PQ, every point on which will be assumed equidistant from O.

Join OP, OQ, CP; draw QS perpendicular to OP, and join PQ, producing it to cut AB produced in T. Then PT will be the tangent at P, and will be perpendicular to CP.

Area of annulus $= 2\pi\, PQ \,.\, PM$;

$$\therefore \text{ potential of annulus} = 2\pi\rho \,.\, \frac{PM.PQ}{OP}$$

$$= 2\pi\rho.\, PQ \sin POC = 2\pi\rho \,.\, \frac{PS}{\cos QPO} \,.\, \sin POC$$

$$= 2\pi\rho \,.\, PS \,.\, \frac{\sin POC}{\sin OPC} = 2\pi\rho \,.\, \frac{a}{f} PS,$$

if $a =$ radius of sphere, and $f = OC$.

Now since OQS is a right-angled triangle, whose vertical angle is exceedingly small, no appreciable error will be committed if we assume $OQ = OS$, and make $PS = OP - OQ$.

Hence the potential of the annulus

$$= 2\pi\rho \frac{a}{f} (OP - OQ).$$

If we add these successive differences for all the annuli of which we may suppose the sphere composed, we have for the potential of the whole shell

$$V = 2\pi\rho \cdot \frac{a}{f} \cdot (OB - OA)$$

$$= \frac{2\pi\rho a}{f} (\overline{a+f} - \overline{a-f})$$

$$= 4\pi\rho a,$$

which is independent of f, and therefore constant for all internal points.

82. We can now deduce the area of the sphere by summing the areas of the elementary annuli. The area of the annulus formed by the revolution of PQ

$$= 2\pi PM \,.\, PQ$$

$$= 2\pi OP \,.\, \sin POC \,.\, PQ$$

$$= 2\pi OP \,.\, \sin POC \,.\, \frac{PS}{\cos QPO}$$

$$= 2\pi \frac{a}{f}\, OP\,(OP - OQ), \text{ by Art. 81,}$$

we may assume without error $2OP = OP + OQ$;

$$\therefore \text{ area of annulus} = \pi \frac{a}{f} (OP^2 - OQ^2)\,;$$

adding up all the annuli, the whole area

$$= \frac{\pi a}{f} (OB^2 - OA^2)$$

$$= \frac{\pi a}{f} (\overline{a+f}|^2 - \overline{a-f}|^2) = 4\pi a^2.$$

Hence if the electrical distribution have a uniform density ρ, the whole quantity

$$Q = 4\pi a^2 \rho\,;$$

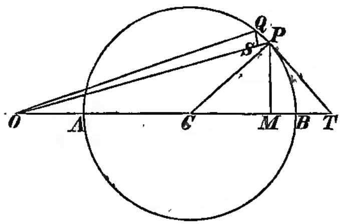
$$\therefore\ V = 4\pi a\rho = \frac{Q}{a},$$

which agrees with the former result.

83. Prop. II. To show that the potential of a uniformly electrified spherical shell at any point without it is the same as if the whole quantity were collected at its centre.

Making the same construction as before, remembering that O is now external, we have by Art. 81,

Fig. 22.



$$\text{potential of annulus} = \frac{2\pi\rho PM\,.\,PQ}{OP}$$

$$= 2\pi\rho\,.\,PQ \sin POC$$

$$= 2\pi\rho\,.\,\frac{PS}{\cos OPQ}\,.\,\sin POC$$

$$= 2\pi\rho PS\,.\,\frac{\sin POC}{\sin OPC} = 2\pi\rho\,\frac{PC}{OC}\,.\,PS$$

$$= \frac{2\pi\rho a}{f}\,(OP - OQ).$$

Summing up the successive differences, we have

$$\text{potential of sphere} = \frac{2\pi\rho a}{f}\,(OB - OA)$$

$$= \frac{2\pi\rho a}{f}\,(\overline{f+a} - \overline{f-a})$$

$$= \frac{4\pi\rho a^2}{f} = \frac{Q}{f},$$

remembering that $Q = 4\pi a^2\rho$.

Hence the potential of the shell is the same as if the whole quantity were collected at its centre.

Cor. 1. It follows that the attraction of a uniformly electrified spherical shell on any external electricity is the same as if the whole quantity on the sphere were collected at its centre. For since the potentials are the same on every external point, the rate of change of potential in any direction must also be the same, and this measures the electrical force, which is consequently the same as if the whole quantity of electricity were accumulated at the centre.

84. Prop. III. The average potential over any sphere in space is the same as the potential at its centre, supposing all electricity external to the sphere. (Gauss.)

By the term 'average potential,' we understand that the sphere's surface is cut up into a large number of equal areas; the average potentials over all the areas added, and the result divided by the sum of the areas. If the areas are not equal, we must multiply each potential by the area over which it is calculated, and divide by the sum of all the areas. We adopt the latter method, and with our usual notation we define the average potential over the sphere by $\frac{\Sigma V\sigma}{\Sigma\sigma}$, where V is the potential at any point, and σ the elementary area over which V is taken. Consider one electrified particle and let its quantity of electricity be denoted by m.

Then, in fig. 22, the potential over the annulus PQ whose area is $2\pi PQ.PM$, due to a quantity m at the point O, may be taken as $\frac{m}{OP}$;

$$\therefore V\sigma = \frac{2\pi PM \,.\, PQ \,.\, m}{OP}$$

$$= 2\pi m \,.\, \frac{PM \,.\, PQ}{OP}.$$

And by last Article,

$2\pi m \,.\, \frac{PM \,.\, PQ}{OP}$ = potential at O of a distribution of density m over the annulus.

Hence summing over the whole sphere

$$\Sigma V\sigma = 2\pi m \Sigma \frac{PM \cdot PQ}{OP}$$

$$= \frac{4\pi a^2 m}{f} \text{ (Prop. II.)},$$

and $$\Sigma\sigma = 4\pi a^2;$$

$$\therefore \frac{\Sigma V\sigma}{\Sigma\sigma} = \frac{m}{f} = \text{potential at centre due to quantity } m \text{ at } O,$$

which proves the proposition as far as a single electrified particle is concerned. In the same way the proposition will be true for any system of electrified particles taken separately, and therefore when added together.

85. Prop. IV. The potential anywhere within an unelectrified conducting sphere is the same as the potential at its centre due to the inducing electrical system.

The potential at the centre is made up of the potential of the inducing system, and of the induced distribution on the sphere. But since the sphere's electrification is only induced, there must be equal amounts of positive and negative electricity equally distant from the centre. Hence the potential due to the induced charge is nil, and the only potential at the centre is that due to the inducing system. But by Art. 63, in every case the potential throughout the sphere is the same as at its centre.

Cor. 1. If the sphere were first raised to a given potential, and then introduced into the electrical system, the potential of the sphere would be raised by the potential at its centre due to the electrical system.

Cor. 2. It also follows that if a sphere be charged with a quantity Q, and placed near a system of external electrified particles, containing quantities $m_1, m_2 \ldots$ of electricity, and at distances $f_1, f_2 \ldots$ from the centre of the sphere, the potential will be given by

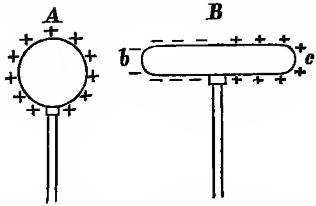
$$V = \frac{Q}{a} + \Sigma \frac{m}{f};$$

and supposing the sphere brought to potential V, the quantity of its electrification is given by

$$Q = a \,.\, V - a\Sigma \frac{m}{f}\,.$$

Cor. 3. This principle may be extended to any conductor, showing that the potential anywhere within the conductor is the potential of an equipotential surface, which cuts the conductor in the line of neutral electrification. Let A be the charged body, and B an unelectrified body near it. The force due to B's electrification at any point within it, is equal and opposite to the force due to that of A. Hence the equipotential surfaces due to B's electrification and to A's, coincide in position, but are not of the same absolute value. Again, considering B's electrification only, it is clear that in passing from any positively to any negatively electrified portion of B, we must pass through a point of zero potential. Hence the surface of zero potential passes through the line of neutral electrification.

Fig. 23.



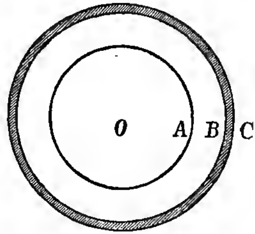
At any point on this surface the whole potential is that due to A, and the potential everywhere within B is the potential of the equipotential surface which passes through the neutral line.

We see therefore that the potential of B is intermediate between the potentials at b and c due to A, and the function of the induced negative charge at b is to keep the potential down, and that of the positive charge at c to keep the potential up to the mean value.

86. Prop. V. To investigate the potential of a system consisting of a sphere and a concentric spherical shell insulated from it, both being charged with known quantities of electricity.

Let O be the common centre, A the sphere charged with a quantity Q of electricity, B the inner and C the outer surface of the spherical shell, which is charged with Q' units of electricity.

Fig. 24.



Now we have to consider not only the distribution of Q on A, and Q' on C, but also the charge induced on B, which will be, by Art. 68, equal and opposite to Q (i.e. $= -Q$), the distribution complementary to this going to the outer surface C and making a quantity $Q + Q'$ on C.

Then potential throughout A is same as potential at O, and

$$= \frac{Q}{OA} - \frac{Q}{OB} + \frac{Q + Q'}{OC}.$$

The potential just outside C is the same as if all the electricity were collected at O and $\therefore = \frac{Q + Q'}{OC}$.

Thus we see that if V_A be the potential within A, and V_C the potential at C,

$$V_A - V_C = \frac{Q}{OA} - \frac{Q}{OB} = \frac{Q}{R} - \frac{Q}{R'}, \text{ suppose.}$$

Again, if $V_A - V_C = 1$, $Q = C$;

$$\therefore C = \frac{RR'}{R' - R}.$$

Thus we see that if $R' - R$ be sufficiently diminished the capacity becomes enormously increased.

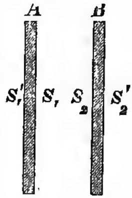
Such an arrangement is called a condenser. The outer coat is generally connected with the earth, and therefore at zero potential, and we have, if V be the potential of the inner coat, and Q the quantity of electricity accumulated,

$$Q = \frac{RR'}{R' - R} \cdot V.$$

If the outer surface be not at zero, the charge $Q + Q'$ is spoken of as the free charge, and in any arrangement in which the inner coat is not completely internal it will also have a free charge.

87. Prop. VI. To find the capacity of a condenser consisting of two parallel plates electrified to given potentials.

Fig. 25.



Let A, B be the two plates, of which A is at potential V_1, and B at potential V_2.

Neglecting a portion of the plates near the edge, we see that we have *three* electrical systems to consider, the outer surfaces S_1' and S_2' of A and B being freely electrified, while the inner surfaces act on each other; these systems being screened from each other by the substance of the conducting plates. We at present consider the bound charge only, produced by the action on each other of the surfaces S_1 and S_2, which are at potentials V_1 and V_2.

Neglecting a portion round the edge, the lines of force which cut both surfaces at right angles are a system of parallel lines: the tubes of force formed by them are cylinders: and in virtue of the relation $F\sigma = \text{constant}$, and $\sigma = \text{a constant}$, F must be constant everywhere between A and B.

Again, the potential between A and B changes from V_1 to V_2.

Hence the rate of change of potential $= \dfrac{V_1 - V_2}{t}$, where t is the distance between S_1 and S_2.

Hence the force anywhere between these surfaces

$$= \frac{V_1 - V_2}{t}. \quad \text{(Art. 37.)}$$

Again, if ρ be the density at any point on an electrified surface, the force just outside $= 4\pi\rho$;

$$\therefore \frac{V_1 - V_2}{t} = 4\pi\rho,$$

$$\text{or } \rho = \frac{V_1 - V_2}{4\pi t};$$

and if S be the area of the surface, $Q = \rho S$;

$$\therefore Q = \frac{(V_1 - V_2) S}{4\pi t}$$

But if $$V_1 - V_2 = 1,\ Q = C;$$
$$\therefore C = \frac{S}{4\pi t}.$$

Again, since tubes of force pass from S_1 to S_2 there must be equal quantities of electricity, of opposite signs, on the two surfaces.

Hence the quantity on $B = -Q = -\dfrac{(V_1 - V_2)S}{4\pi t}$.

If the dielectric between S_1 and S_2 be not air but glass, or shellac, a substance whose inductive capacity is represented by K,

$$C = \frac{KS}{4\pi t} = \frac{S}{4\pi \frac{t}{K}} = \frac{S}{4\pi t'}, \quad \text{(Art. 76,)}$$

where $t' = \dfrac{t}{K}$ = thickness 'reduced to air.'

To complete the investigation we ought to find the amount of the free charge on the two surfaces S_1' and S_2'; this can only be done in a few particular cases. If however their capacities be C_1 and C_2, the quantities of the free charges will be C_1V_1 and C_2V_2 respectively.

The same theory can be applied to every form of condenser, provided the thickness be small and the two surfaces everywhere parallel.

In the common form of Leyden jar, where the outer coat is connected with the earth, and the inner coat is nearly a closed surface, the free charge is only the charge of the knob and wire, which project from the inner coat and are used for charging it.

88. Prop. VII. To find the attraction between the opposite plates of the condenser in the last Article.

Let A be a movable and B a fixed plate.

Then if ρ be the density at any point on A,

$$\frac{V_1 - V_2}{t} = 4\pi\rho;$$

and since the density on B is $-\rho$, the force produced by B near an element of A is $2\pi\rho$. (Art. 31.)

If S be the area of A, and the force on A be denoted by F,

$$F = 2\pi\rho \times \rho S = 2\pi\rho^2 S$$

$$= 2\pi S\left(\frac{V_1 - V_2}{4\pi t}\right)^2 = \frac{S}{8\pi t^2}(V_1 - V_2)^2,$$

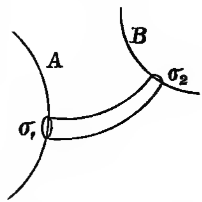
and for the difference of potential deduced from an observed force

$$V_1 - V_2 = t\sqrt{\frac{8\pi F}{S}},$$

the formula employed in absolute and attracted disc-electrometers.

89. Cor. It follows that the force between any two surfaces between which lines of force can be drawn, will be proportional to the square of the difference of potential. For the two surfaces A and B can be divided into sets of elements σ_1, σ_2, which form the opposite ends of tubes of force. The lines of force, and therefore the elements σ_1, σ_2, will not be altered by changes in potentials of A and B, provided they remain unequal*.

Fig. 26.



Although through the tube of force the force is not uniform, its value at any point will depend on the difference of potential of A and B, and be represented near to σ_1 by $\kappa(V_1 - V_2)$;

$$\therefore \kappa(V_1 - V_2) = 4\pi\rho_1,$$

if ρ_1 be the density on σ_1.

Again the force $4\pi\rho_1$ just outside A is made up of a force due to B and another due to A. Let these parts be represented by the fractions λ and λ' respectively. Hence the force at any point σ_1 is $4\lambda'\pi\rho_1$; and the whole quantity is $\rho_1\sigma_1$;

$$\therefore \text{Force on element } \sigma_1 = 4\lambda'\pi\rho_1^2\sigma_1 = 4\lambda'\pi\sigma_1\left\{\frac{\kappa(V_1 - V_2)}{4\pi}\right\}^2.$$

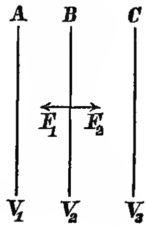
Thus the force on each element varies as $(V_1 - V_2)^2$, and the resultant of all such elementary forces will therefore also vary as $(V_1 - V_2)^2$.

* It should be noticed that this proposition is only true when one surface entirely encloses the other, as otherwise all lines of force do not go from A to B.

90. Prop. VIII. Two fixed plates are kept at potentials V_1 and V_3 and a third movable plate kept at potential V_2 is placed symmetrically between them, to find the resultant force on the middle plate.

Supposing V_2 to be greater than V_1 and V_3, and $V_1 > V_3$.

Fig. 27.



By the last proposition for the attraction of A on B,

$$F_1 = \frac{S}{8\pi t^2}(V_2 - V_1)^2, \text{ where } S = \text{area of } B:$$

also the force exerted by C on B,

$$F_2 = \frac{S}{8\pi t^2}(V_2 - V_3)^2.$$

Hence resultant force towards C, the plate of lower potential,

$$= F_2 - F_1 = \frac{S}{8\pi t^2}\{(V_2 - V_3)^2 - (V_2 - V_1)^2\}$$

$$= \frac{S}{4\pi t^2}\left(V_2 - \frac{V_1 + V_3}{2}\right)(V_1 - V_3),$$

the formula which is used in the quadrant electrometer, and which the corollary to the last proposition shows to be true whatever be the form of A and C, supposing them to be symmetrical, making a proper modification of the constant multiplier. This constant is in practice determined for each quadrant electrometer by comparing it with an absolute electrometer.

91. Prop. IX. To calculate the energy of the discharge of a Leyden jar.

Referring to Prop. VI. we see that we have a quantity $\frac{(V_1 - V_2)S}{4\pi t}$ at potential V_1, and a quantity $-\frac{(V_1 - V_2)S}{4\pi t}$ at potential V_2. Hence (Art. 78) the whole energy of the bound charge

$$= \frac{1}{2}\frac{(V_1 - V_2)S}{4\pi t}V_1 - \frac{1}{2}\frac{(V_1 - V_2)S}{4\pi t}V_2$$

$$= \frac{1}{2}\frac{(V_1 - V_2)^2 S}{4\pi t} = \frac{1}{2}Q(V_1 - V_2).$$

This is the formula generally used, but we must add to this the energy of discharge of the two free charges if the jar be completely discharged. This will be $\frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2$, a small quantity which in practice may be neglected.

This formula also supplies us with the heating power of the discharge, since when work is done in no other form by an electrical discharge its energy is converted into heat. To give the result in absolute thermal units we must divide by Joule's mechanical equivalent of heat.

92. The whole energy expressed in the above formula can never be obtained in practice when the dielectric is different from air, owing to apparent absorption by the dielectric of part of the charge. Although this portion of the charge can be regained as the residual charge it is obvious there will be a loss of energy when the discharge takes place in two portions instead of all at once. In fact, if V be the potential and Q the quantity of the first discharge, and v the potential and q the quantity of the residual discharge, the energy obtained on the double discharge $= \frac{1}{2}(VQ + vq)$, while the whole energy is $\frac{1}{2}(V + v)(Q + q)$.

93. Prop. X. A Leyden jar having capacity for bound charge C has an inner coat whose free capacity is C_1 and an outer coat whose free capacity is C_2. The jar is charged to potential V and insulated and the knob is then connected with the ground. To find the potential of the outer coat and the charge of the jar.

Let the quantity of electricity on inner coat

$$= Q_1 = (C + C_1) V,$$

and the quantity on outer coat $= Q_2 = - CV$.

When the inner coat is connected with the ground the charge of the outer coat is divided between bound and free charge in the ratio $C : C_2$.

$\therefore$ Bound charge of outer coat

$$= \frac{C}{C + C_2} Q_2 = - \frac{C^2}{(C + C_1)(C + C_2)} Q_1,$$

Free charge of outer coat $= \dfrac{C_2}{C + C_2} Q_2$;

$\therefore$ Potential of outer coat $= \dfrac{Q_2}{C+C_2} = -\dfrac{CV}{C+C_2}$.

Also the amount of electricity on inner coat

$$= \frac{C^2}{(C+C_1)(C+C_2)} Q_1;$$

$$\therefore \text{ Loss} = \left\{1 - \frac{C^2}{(C+C_1)(C+C_2)}\right\} Q_1.$$

94. **Prop. XI. A Leyden jar is charged and insulated. Successive contacts are made with the inner and outer coats. Find the amount of electricity removed by n contacts with inner or outer coat.**

Using the notation of the preceding Article, and writing $\dfrac{C}{C+C_1} = m$ and $\dfrac{C}{C+C_2} = m'$, we see that when the outer coat is to earth, the charge on inner coat is divided in the ratio $m : 1-m$ between bound and free charge. Also when the knob is to earth the charge on outer coat is divided in the ratio $m' : 1-m'$. We see therefore

At *first contact with knob:*

Free charge on outer coat $= (1-m')\,Q_2$,
Bound charge.............. $= m'Q_2 = -\,mm'Q_1$.

At *first contact with outer coat:*

Free charge on inner coat $= (1-m)\,mm'Q_1$,
Bound charge.............. $= m^2m'Q_1 = -\,mm'Q_2$.

At *second contact with knob:*

Free charge on outer coat $= (1-m')\,mm'Q_2$,
Bound...................... $= mm'^2Q_1 = -\,m^2m'^2Q_2$.

By similar reasoning *after* n *contacts with knob:*

Free charge on outer coat $= (1-m')\,m^{n-1}m'^{n-2}Q_2$,
Bound...................... $= m^{n-1}m'^nQ_2 = -\,m^nm'^nQ_1$.

Hence the amount removed by n contacts with knob

$$= Q_1(1 - m^nm'^n),$$

and quantity removed by n contacts with outer coat

$$= Q_2(1 - m^nm'^n).$$

But generally C_1 and C_2 are very small compared with C, and therefore m, m' fractions very near unity. Hence we see that a large fraction of the charge remains after numerous contacts, and it would require an infinite number of contacts to discharge the jar.

95. Prop. XII. Two Leyden jars are charged to different potentials and afterwards have their knobs brought into contact, the outer coats being kept in connection with the earth. To find the potential of each jar after contact.

Let C_1, C_2 be the capacities, and V_1, V_2 the potentials of the jars, and let V be their common potential after contact. Then since the whole amount on the inner coats is unaltered

$$(C_1 + C_2)\, V = C_1 V_1 + C_2 V_2;$$

$$\therefore\ V = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2},$$

an equation for V.

COR. It follows that there will always be a loss of energy when two jars at different potentials are united.

For energy before contact $= \frac{1}{2}(C_1 V_1^2 + C_2 V_2^2)$,

and energy after contact $= \frac{1}{2}(C_1 + C_2)\left(\frac{C_1 V_1 + C_2 V_2}{C_1 + C_2}\right)^2$.

Now $(C_1 V_1^2 + C_2 V_2^2)(C_1 + C_2) >$ or $< (C_1 V_1 + C_2 V_2)^2$,

as $C_1 C_2 (V_1^2 + V_2^2) >$ or $< 2C_1 C_2 V_1 V_2$,

or as $(V_1 - V_2)^2 >$ or < 0;

the left-hand side is obviously the greater, and hence the sum of the energies of the separate jars is greater than that of the two combined.

96. It was by a particular experimental application of the above that Faraday determined the specific inductive capacity of different substances. He constructed two exactly similar Leyden jars, the coatings of which were so arranged that the dielectric could be changed at pleasure. One of the jars had air for its dielectric, and the other the substance to be experimented upon.

Let now K be the unknown specific inductive capacity; then if C be the capacity of the jar with air, CK is the capacity with other substance as dielectric.

Let now the jar with air be raised to potential V. On dividing the charge with the other jar, which is uncharged, the potential V' evidently becomes

$$V' = \frac{CV}{C(1+K)} = \frac{V}{1+K};$$

$$\therefore\ V'(1+K) = V,$$

or

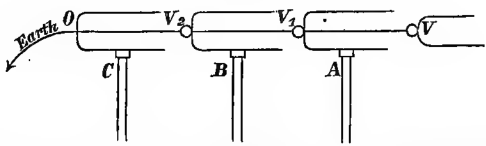
$$K = \frac{V-V'}{V'}.$$

Now V and V' are determined by experiment, after the operations indicated, and the value of K for the substance in question becomes known.

97. Prop. XIII. To show that the whole charge in a battery of similar jars charged by cascade only equals the charge of a single jar.

Let A, B, C be such a series of jars of which the first is brought up to potential V, and the last is to earth.

Fig. 28.



Let the potential of the knob of B and of the outer coating of A be V_1.

Let the potential of the knob of C and of the outer coating of B be V_2.

And let C be the capacity of each jar.

Then the *equal* quantities in the three jars are

$$C(V-V_1),\ C(V_1-V_2),\ CV_2.$$

Hence the whole charge in the battery

$$= C(V-V_1) + C(V_1-V_2) + CV_2$$
$$= CV$$

= charge of single jar charged alone to the same potential.

The same method applies to any number of jars.

98. Prop. XIV. To find the work done by a conducting plate communicating with the earth which is allowed to move up parallel to another equal plate which is kept at constant potential.

The principle of Art. 78 might here be employed, since it is clear the work done is stored up in the form of electrical energy, and the energy obtained on discharging the condenser is the equivalent of the work so stored up.

In this case we give an independent investigation.

Let A be the plate kept at constant potential, and B that connected with the earth.

Fig. 29.

By Prop. VII. the force on $B = \dfrac{V^2 S}{8\pi OP^2}$.

The average force through the element PQ is $\dfrac{V^2 S}{8\pi \,.\, OP \,.\, OQ}$.

Hence the work done by the plate in coming up from P to Q

$$= \frac{V^2 S \,.\, PQ}{8\pi \,.\, OP \,.\, OQ}$$

$$= \frac{V^2 S}{8\pi} \,.\, \frac{OP - OQ}{OP \,.\, OQ} = \frac{V^2 S}{8\pi}\left(\frac{1}{OQ} - \frac{1}{OP}\right).$$

Adding up the work done on successive elements of the path we see that if t be the ultimate distance,

$$\text{whole work} = \frac{V^2 S}{8\pi} \,.\, \frac{1}{t} = \tfrac{1}{2} Q V,$$

as might have been anticipated.

99. We now give an example in which the work done is deduced from the change in energy of the system.

Prop. XV. Two plates are placed parallel to each other, charged as a condenser, insulated, and separated to an infinite distance. To find the work done in the removal.

Let C be the capacity of the bound charge, and C' the capacity of free charge in condenser. The free capacity

varies with every new position of the plates; we shall however assume here that when both sides of the plates are electrified freely the capacity of the free charge becomes doubled, i.e. when the plates are entirely removed from each other's influence the capacity of each becomes $2C'$.

This assumption would be correct supposing the two plates to begin with were indefinitely near together, since in that case their external surfaces would be electrified as a single plate; but after separation they would be electrified as two separate plates, each of the same size as the former. We can therefore only assume the result as more than approximately true in practice when the distance of the plates is very small.

After charging the amount on the positive plate is

$$(C+C')\,V,$$

and its energy of discharge is $\frac{1}{2}V^2(C+C')$.

The quantity on the negative plate is

$$-CV,$$

and its potential zero.

Hence the energy of the whole system

$$=\tfrac{1}{2}V^2(C+C').$$

After removal the quantity on each plate is unaltered, but the capacity, as assumed above, is $2C'$.

Hence potential of positive plate $= \dfrac{C+C'}{2C'}V$,

and the energy of its discharge $= \dfrac{(C+C')^2V^2}{4C'}$;

and the energy of negative plate $= \dfrac{C^2V^2}{4C'}$.

Hence gain of energy

$$= \frac{(C+C')^2V^2 + C^2V^2}{4C'} - \frac{(C+C')\,V^2}{2}$$

$$= \frac{V^2}{4C'}\{(C+C')^2 + C^2 - 2C'(C+C')\}$$

$$= \frac{V^2}{4C'}(2C^2 - C'^2)$$

= work done in separation of plates.

100. The next two propositions will be found useful in considering some cases of induced electricity.

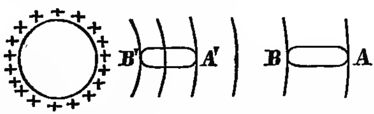
DEF. A SYSTEM OF EQUIPOTENTIAL SURFACES *denotes a system of surfaces such that the difference of potential between any two consecutive surfaces of the system is constant.*

101. Prop. XVI. In a system of equipotential surfaces the distance of consecutive surfaces increases as the potential diminishes.

Let d be the distance of two consecutive surfaces measured along a line of force, and F the *average* value of the force in direction of a line of force. Then between each successive pair of surfaces Fd is constant, and it is clear that as F diminishes d increases; or the distance of consecutive surfaces grows greater as the force grows less. Now it is clear that force and potential increase or diminish together as we draw nearer to or recede further from attracting matter, and the distance of consecutive surfaces consequently increases as the potential diminishes.

COR. There will be an induced current in any conductor moving near an electrified system. For draw any system of equipotential surfaces, and let a conductor move from the position AB to $A'B'$.

Fig. 30.



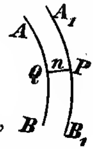
It is clear that the difference of potential between A and B is one unit, while that between $A'B'$ is two units. Hence as the body moves into regions of greater potential, the difference of potential of its ends constantly increases, and to equalize this increasing inequality, a flow of electricity follows in the direction for + electricity from B to A, as long as the movement across equipotential surfaces lasts.

102. Prop. XVII. To calculate the rate of motion of any point on an equipotential surface of given value as the electrification of the system proceeds.

Let AB be an equipotential surface of value V when the charge of the system is M, and let $A'B'$ be the equipotential

surface of same value when the system has received a small increase q.

Fig. 31.



If m be an element of the electrification distant r from Q, the potential at the point Q

$$= \Sigma \frac{m}{r} = V.$$

If now the charge M receive a small increment q, the density at each point alters in ratio $\frac{q}{M}$. Hence the potential at Q rises by $\frac{q}{M} V$.

But the potential at P is now V.

Hence the work done on a + unit carried from P to Q

$$= \frac{q}{M} V,$$

and this also equals Fn, where F is the resultant force, and n the length PQ;

$$\therefore Fn = \frac{q}{M} V,$$

$$\text{or } n = \frac{V}{MF} q.$$

Hence the rate of motion of the surface at P varies jointly as the potential and the rate of electrification, and varies inversely jointly as the whole charge and the force at the point.

Cor. 1. Since V is represented by $\Sigma \frac{q}{r}$, and F by $\Sigma \frac{q}{r^2} \cos\phi$, where ϕ is the angle between the normal to the surface and r, it is clear that $\frac{V}{F}$ or

$$\frac{\Sigma \frac{q}{r}}{\Sigma \frac{q}{r^2} \cos\phi}$$

will on the whole increase as r on the whole increases, i.e. as V diminishes. Hence, on the whole, the lower equipotential

surfaces move during electrification more rapidly than the surfaces at which the potential has a higher value.

COR. 2. It follows from the last Cor. that there will be an induced current in a conductor during the electrification of any neighbouring conductor. For in fig. 30 the potential surfaces at A are moving more rapidly than those at B. Hence the difference of potentials at A and B is constantly increasing, or B's potential relatively to A constantly rising, which determines a flow of electricity from B to A.

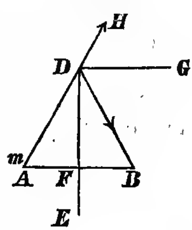
103. We have already shown, Art. 72, that if we have an equipotential conducting surface passing between two parts of an electrical system, we may substitute a free electrification of that surface for either part of the system as far as actions on the opposite side of the surface are concerned. One of these systems is then called the electrical image of the other. The following formal definition is due to Prof. Clerk Maxwell.

DEF. "AN ELECTRICAL IMAGE *is a point or system of "points on one side of an electrified surface which produces " on the other side of that surface the same electrical action "as the actual electrification does produce.*"

104. To illustrate the method, suppose we are required to find the electrical force at any point due to a system consisting of an electrified point, and an infinite conducting plane connected with the earth.

Let A be the electrified point containing m units of electricity, and DE the conducting plate. Draw AF perpendicular to DE, and produce it to B, so that $FB = FA = p$. If we imagine a quantity $-m$ of electricity at B, it is clear that for the system m at A and $-m$ at B, DE will be a surface at zero-potential. For the potential at any point D is $\frac{m}{DA} - \frac{m}{DB} = 0$; because $DB = DA$. Hence we may substitute for the electrification of the plate the charged point B, as far as places to the left of DE are concerned.

Fig. 32.



To find the density at any point on the plate, we have

only to find the resultant force due to the electrified points A and B, and divide by 4π.

This resultant will be normal to the plate, and consists of two components, one repulsive along AD, and one attractive along DB. Hence,

$$\text{Resultant force} = \frac{m}{BD^2}\cos BDG + \frac{m}{AD^2}\cos HDG$$

$$= \frac{2m}{AD^2}\cos DAF = \frac{2m}{AD^2}\cdot\frac{AF}{AD} = \frac{2mp}{AD^3},$$

and is directed to the right of DE. Hence the density will be negative, and at point D

$$= -\frac{2mp}{4\pi AD^3} = -\frac{mp}{2\pi AD^3},$$

or the density varies inversely as the cube of the distance from the electrified point.

105. We may deduce next the electrical influence of a point on a sphere by considering two points having charges e_1 and $-e_2$ of electricity placed at points A, B. We shall have for the potential at any point distant r_1 and r_2 from A, B respectively

$$\frac{e_1}{r_1} - \frac{e_2}{r_2}.$$

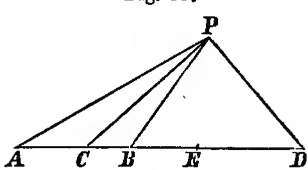
Hence for surface of zero-potential

$$\frac{e_1}{r_1} = \frac{e_2}{r_2}, \text{ or } r_1 : r_2 :: e_1 : e_2.$$

Or, the distances of any point on the surface from A and B are in the constant ratio e_1 to e_2.

We can easily show that the locus of a point satisfying this condition is a sphere.

Fig. 33.



For let P be a point on the locus, and e_1 greater numerically than e_2. Divide AB internally and externally in CD, so that

$$AC : CB :: AD : DB :: e_1 : e_2 \ldots\ldots\ldots\ldots\text{(i)}.$$

Hence $AP : PB :: AC : CB$; $\therefore$ PC bisects APB,

and $AP : PB :: AD : DB$; $\therefore$ PD bisects APB externally;

$\therefore$ CPD is a right angle.

Hence the locus of P on the plane of the paper will be a circle whose diameter is CD: and the same property will be true for each point on the sphere whose diameter is CD.

106. To determine the position and dimensions of the sphere we have the following relations.

From (i) we have

$$AC + CB : CB :: e_1 + e_2 : e_2;$$

$$\therefore \quad CB = \frac{e_2}{e_1+e_2} AB \text{ and } AC = \frac{e_1}{e_1+e_2} AB \ldots\ldots \text{(ii)}.$$

Again from (i),

$$AD - DB : DB :: e_1 - e_2 : e_2,$$

$$\therefore \quad DB = \frac{e_2}{e_1 - e_2} AB \text{ and } AD = \frac{e_1}{e_1 - e_2} AB \ldots\ldots \text{(iii)}.$$

Also $CD = CB + BD = \dfrac{e_2}{e_1+e_2} AB + \dfrac{e_2}{e_1 - e_2} AB = \dfrac{2e_1e_2}{e_1^2 - e_2^2} . AB.$

Hence radius of circle $= \dfrac{e_1e_2}{e_1^2 - e_2^2} . AB \ldots\ldots\ldots\ldots$ (iv).

And $EB = CE - CB = \dfrac{e_1e_2}{e_1^2 - e_2^2} . AB - \dfrac{e_2}{e_1+e_2} AB = \dfrac{e_2^2}{e_1^2 - e_2^2} AB$

$$= \frac{e_2}{e_1} EC.$$

Similarly $\qquad EA = \dfrac{e_1}{e_2} EC;$

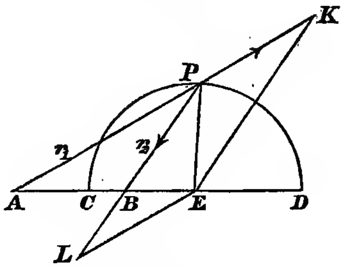
$$\therefore \ EB . EA = EC^2.$$

A and B are called conjugate points with reference to the sphere.

Hence we see that for either e_1 at A, or $-e_2$ at B, we may, as far as actions on the opposite side of the sphere are concerned, substitute the electrified surface; and, conversely, for the electrified surface, we may substitute these points.

107. To find the density at any point, we have to find the resultant force at any point on the spherical surface, and as before divide it by 4π.

Fig. 34.



The forces on P are clearly $\frac{e_1}{r_1^2}$ along AP, and $\frac{e_2}{r_2^2}$ along PB, and their resultant is along PE, since it is normal to the equipotential surface. Let F be the magnitude of the resultant, then completing the parallelogram $EKPL$,

$$F : \text{component in } AP :: PE : PK,$$

$$\text{or } F = \frac{PE}{PK}\frac{e_1}{r_1^2},$$

and by similar triangles

$$PK : BE :: AP : AB,$$

$$\therefore PK = \frac{BE}{AB} \cdot AP,$$

substituting $\quad F = \frac{AB \cdot PE}{BE} \cdot \frac{e_1}{r_1^3} = a \cdot \frac{e_1^2 - e_2^2}{e_2^2} \cdot \frac{e_1}{r_1^3},$

if a be the radius of the sphere.

Since this resultant is inwards, we must express the density by

$$-\frac{e_1(e_1^2 - e_2^2)}{4\pi e_2^2} \cdot \frac{a}{r_1^3},$$

and we conclude that a distribution whose density is given by the above law produces within the sphere a force equal and opposite to e_1 at A, and without the sphere a force equal to $-e_2$ at B, and is therefore the distribution produced by e_1 placed at A.

Again, if the density at P be expressed by

$$+\frac{e_2(e_1^2-e_2^2)}{e_1^2}\frac{a}{r_2^3},$$

we conclude that this distribution produces without the sphere a force equal and opposite to that of $-e_2$ at B, and within the sphere a force equal and opposite to that of e_1 at A.

The whole quantity of the distribution being in either case $\pm e_2$.

(i) In the *first case* let $AE=f$, then $BE=\frac{a^2}{f}$,

$$\text{and } f=\frac{e_1}{e_2}a.$$

Hence the law of density becomes, on substitution for e_2 and reduction,

$$-\frac{e_1(f^2-a^2)}{4\pi a\,.\,r_1^3}.$$

(ii) In the *second case* let $BE=f'$, then $AE=\frac{a^2}{f'}$,

$$\text{and } f'=\frac{e_2}{e_1}a.$$

Hence substituting for e_1 the law of density is

$$+\frac{e_2(a^2-f'^2)}{4\pi ar_2^3}.$$

We see now that we can include both cases in the following statement:

If there be taken on the radius of a sphere two conjugate points, and a quantity of electricity e be placed at one of them whose distance from the centre is f, it will induce a distribution over the sphere whose law of density is

$$-\frac{(a^2\sim f^2)\,e}{4\pi ar^3},$$

where r = distance from the electrified point
and a = radius of sphere,

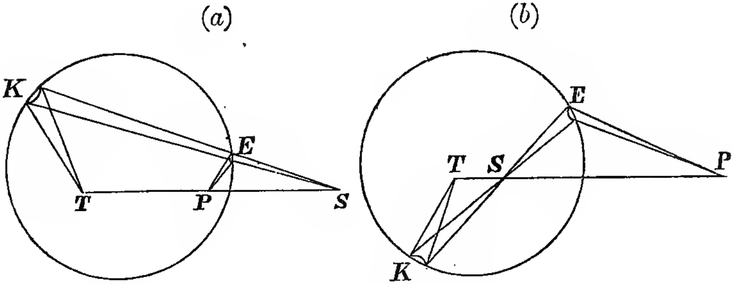
and the resultant effect of this distribution on all points on the same side of the spherical surface is the same as of a particle at the conjugate point having a charge represented by $-\frac{a}{f}e$; the whole quantity of the distribution being, when the electrified point is within the sphere, $-e$, and when without the sphere, $-\frac{a}{f}e$.

108. The following direct geometrical proof of the proposition of the preceding article is a modification of that originally given by Sir William Thomson.

Prop. XVIII. A distribution of matter is made over a spherical surface whose density at any point varies inversely as the cube of its distance from a fixed point, show that the potential of the distribution at any point on the opposite side of the spherical surface is the same as that due to a certain quantity of matter at the given fixed point.

Suppose S the given fixed point in Fig. (a) external, and in Fig. (b) internal, and let P be a point on the opposite side of the spherical surface on which we shall estimate the potential;

Fig. 35.



in Fig. (a) the distance of S from the centre f, and in Fig. (b) f'.

Join SP and produce it to T so that

$$\text{Fig. }(a)\quad SP \,.\, ST = f^2 - a^2,$$
$$\text{Fig. }(b)\quad SP \,.\, ST = a^2 - f'^2.$$

Through S draw any line meeting the sphere in E and K, join PE and TK;

$$\therefore\ SE \,.\, SK = SP \,.\, ST \text{ in both figures;}$$

$$\therefore \text{ the triangles } SEP,\ STK \text{ are similar.}$$

Conceive now the line KES to move so as to trace out a small cone whose vertex is S, and which cuts the spherical surface in elementary areas E and K. It is clear that E and K are corresponding elements, so that the whole surface is exhausted *simultaneously* by a series of elements E and K.

Now the potential at the point P due to the element E of the distribution,

$$= \frac{\text{mass of } E}{EP},$$

and density at E $\qquad = \dfrac{\lambda}{SE^3}$;

$$\therefore \text{ potential due to } E = \frac{\lambda E}{EP \,.\, SE^3}.$$

Again, since the tangent planes at E, K are equally inclined to SEK,

$$E : K :: SE^2 : SK^2;$$

$$\therefore \text{ potential due to } E = \frac{\lambda K}{EP \,.\, SE \,.\, SK^2} = \frac{\lambda K}{EP \,.\, SK\,(f^2 \sim a^2)};$$

also by similar triangles $EP : SP :: TK : SK$,

$$\therefore\ EP \,.\, SK = SP \,.\, TK,$$

$$\therefore \text{ potential due to } E = \frac{\lambda K}{SP\,(f^2 - a^2) \,.\, TK};$$

$$\therefore \text{ potential of sphere} = \frac{1}{SP(f^2 - a^2)} \,.\, \Sigma \frac{\lambda K}{TK}.$$

But $\Sigma \dfrac{\lambda K}{TK}$ represents the potential at T of a uniform dis-

tribution whose density is λ, which by Prop. 1 is $4\pi\lambda a$, since T is necessarily an internal point.

$$\text{Hence the potential of the sphere on } P = \frac{4\pi\lambda a}{(f^2 \sim a^2)\, SP}$$

$$= \text{potential on } P \text{ of a mass } \frac{4\pi\lambda a}{f^2 \sim a^2} \text{ at } S,$$

or substituting electric distribution for matter, and reversing the sign of the distribution on the sphere,

Potential due to quantity $\dfrac{4\pi\lambda a}{f^2 \sim a^2}$ at S + Potential due to the distribution of density $-\dfrac{\lambda}{SE^3} = 0$, on any point P on the opposite side of the surface to S.

And this is the condition which must be satisfied by the distribution induced by an electrified particle at S (Art. 63).

If we put $\qquad m = \dfrac{4\pi\lambda a}{f^2 \sim a^2}$, we get

Potential due to m at S + Potential due to distribution of density $-\dfrac{(f^2 \sim a^2)\, m}{4\pi a \,.\, SE^3} = 0$, at any point on the opposite side of the surface.

To find the whole quantity distributed over the sphere we see

$$\text{quantity on element } E = \frac{\lambda E}{SE^3} = \frac{\lambda K}{SE \,.\, SK^2} = \frac{\lambda K}{(f^2 \sim a^2)\, SK};$$

$$\therefore \text{quantity distributed} = \frac{1}{f^2 \sim a^2} \Sigma \frac{\lambda K}{SK},$$

and $\Sigma \dfrac{\lambda K}{SK}$ = potential at S of a uniform distribution of density λ.

In figure (a) S is external,

$$\therefore \Sigma \frac{\lambda K}{SK} = \lambda \frac{4\pi a^2}{f},$$

and the whole amount of distribution $= \dfrac{\lambda 4\pi a^2}{f(f^2 - a^2)}$

$$= \frac{a}{f} m.$$

In figure (*b*) S is internal, and

$$\therefore \Sigma \frac{\lambda K}{SK} = \lambda \,.\, 4\pi a,$$

$$\therefore \text{ whole amount of distribution} = \frac{\lambda \,.\, 4\pi a}{a^2 - f^2}$$

$$= m,$$

as has been already shown.

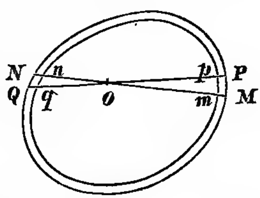
By choosing conjugate points, it is easy to show that the two distributions, one derived from the external and the other from the internal point, are identical, and the proposition of Art. 107 follows immediately.

109. In the next few Articles we give the investigation of two cases of electrification very useful in practice; the first that of a thin circular plate, the second that of a very thin and long cylinder. We may regard both these cases as the limiting form of a spheroid, which in the first case becomes extremely oblate, and in the second case extremely prolate. We require therefore the following preliminary proposition.

Prop. XIX. To show that the attraction of a homogeneous shell bounded by two similar and similarly situated spheroids on an internal point vanishes.

Let O be the internal point, and let any cone of small vertical angle cut off from the bounding surfaces the element Pm and Nq.

Fig. 36.



Since any section of the shell consists of two similar ellipses, the same diameter will bisect PQ and pq; $\therefore Pp = Qq$, and similarly $Mm = Nn$.

Hence the two small frusta Pm, Qn are of the same thickness, and since the vertical angle is small, their masses are proportional

to OP^2 and OQ^2, and their attractions on O are inversely in this ratio, and are therefore equal and in opposite directions. The same is true for each pair of small elements similarly described, and the whole shell consequently exerts no attraction on O.

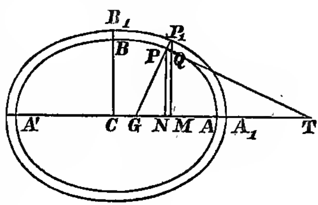
This proof is derived from Todhunter's *Analytical Statics.*

110. Prop. XX. To find the law of density on a freely electrified spheroid.

By the last proposition the law of density is the same as the law of thickness of a very thin material shell bounded by two similar and similarly situated spheroids.

Let AA' be the major axis of generating ellipse, and P, P_1 be the thickness of the shell at the point P, then PP_1 produced is the normal to the ellipse at P. Draw PN, P_1QM perpendicular to AA'. Join PQ, and produce it to T, then PQT is the tangent to the ellipse at P, and P_1Q are corresponding points on the two similar ellipses.

Fig. 37.



Then we have

$$PP_1 = P_1Q \cos PP_1Q = P_1Q \cos GPN = P_1Q \cdot \frac{PN}{PG}.$$

Again $$P_1M^2 = \frac{b'^2}{a'^2}(a'^2 - CM^2),$$

where a', b' = the semi-major and minor axes of outer ellipse,

and $$QM^2 = \frac{b^2}{a^2}(a^2 - CM^2).$$

On subtracting, since $$\frac{b^2}{a^2} = \frac{b'^2}{a'^2},$$

$$P_1M^2 - QM^2 = \frac{b^2}{a^2}(a'^2 - a^2) = b'^2 - b^2;$$

$$\therefore\ P_1Q\,(P_1M + QM) = (b' + b)\,(b' - b);$$

$$\therefore\ PN \,.\, PQ = b\,(b' - b),$$

remembering that the shell is indefinitely thin.

Hence thickness at $P = \frac{b(b'-b)}{PG}$.

If we substitute an electrical distribution and if ρ_0 represent the density at B the extremity of minor axis,

$$\text{density at } P = \frac{b\rho_0}{PG} = \frac{a\rho_0}{\sqrt{a^2 - e^2x^2}},$$

where $x = CM$, and $e =$ eccentricity of ellipse.

To find the whole quantity of the electrification, the quantity on an element formed by revolution of PQ round the minor axis

$$= 2\pi CN.PQ.\frac{b\rho_0}{PG} = 2\pi b\rho_0.\frac{CN.PQ}{PG}.$$

But $CG = e^2CN$; $\therefore CN = \frac{NG}{1-e^2}$;

$$\therefore \text{ quantity on elementary annulus} = \frac{2\pi b\rho_0}{1-e^2}.\frac{NG.PQ}{PG}$$

$$= \frac{2\pi b\rho_0}{1-e^2}.PQ \sin GPN$$

$$= \frac{2\pi b\rho_0}{1-e^2}(PN - QM),$$

adding the successive values of this difference,

$$\text{whole amount of electricity} = \frac{2\pi b\rho_0.2b}{1-e^2} = 4\pi a^2\rho_0.$$

111. To deduce the electrification of a circular plate, we notice that a thin circular plate is the limiting form of an oblate spheroid whose minor axis is indefinitely short, while the major axis is finite and equal to a.

In virtue of the relation $b^2 = a^2(1-e^2)$, we see that in this case we must make $e = 1$.

Hence for the density at any point in terms of the density ρ_0 at the centre, we have

$$\frac{a\rho_0}{\sqrt{a^2 - x^2}} = \rho,$$

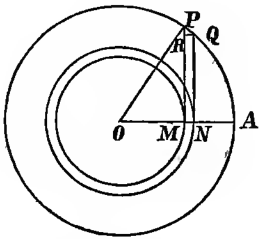
where $a =$ radius, and $x =$ distance of point from the centre.

Also the whole quantity is $4\pi a^2\rho_0$, which shows that the average density is twice the density at the centre.

112. To calculate the potential of a thin circular plate freely electrified, we have only to calculate the potential at the centre of a distribution whose law is given above.

Conceive the plate cut up into a very large number of

Fig. 38.



narrow annuli, and let OA cut one in M, N. Draw MP, NQ perpendicular to OA, and QR perpendicular to PM, and join OP.

Then the potential at O of the ring

$$= \frac{2\pi OM \,.\, MN}{OM} \cdot \frac{a\rho_0}{\sqrt{OP^2 - OM^2}} = 2\pi a\rho_0 \,.\, \frac{MN}{PM}$$

$$= 2\pi a\rho_0 \,.\, \frac{PQ \sin QPR}{PM} = 2\pi a\rho_0 \,.\, \frac{PQ \cos OPM}{PM}$$

$$= 2\pi a\rho_0 \,.\, \frac{PQ}{a} = 2\pi a\rho_0 \,.\, \angle POQ,$$

since $\dfrac{PQ}{a}$ is the circular measure of the angle POQ.

Hence adding all the small angles corresponding to successive annuli, we have clearly

$$\text{Potential} = 2\pi a\rho_0 \,.\, \frac{\pi}{2} = \pi^2 a\rho_0.$$

But we have only yet considered one surface of the plate which will be electrified on both surfaces.

Hence Potential of Plate $= 2\pi^2 a\rho_0$,

$$\text{or} \quad V = \frac{\pi}{2a} . Q,$$

when $V = 1$, $Q = C$,

$$\therefore C = \frac{2a}{\pi}.$$

Hence capacity of plate : capacity of sphere of same radi
$:: 2 : \pi$.

If however the plate forms one conductor of a condens
the capacity will only be $\frac{a}{\pi}$, since the free charge exists
one side only (see Art. 99).

113. Prop. XXI. To find the capacity of a very long and th cylinder.

It is clear that, neglecting the ends, the electrification the cylinder will be sensibly of uniform density. We mig see this by viewing the cylinder as the limit of a very prola spheroid, whose major axis is very large compared to i minor axis. We shall assume the potential everywhe within the cylinder the same as at the middle point of i axis.

Let O be the middle of the axis, and a the radius of tl cylinder. Taking an element PQ round the cylinder, the quantity of electricity on its surface is

Fig. 39.

$$2\pi . a\,PQ . \rho,$$

and the potential at O of this element

$$= \frac{2\pi a\rho PQ}{OQ}.$$

Draw OC perpendicular to the surface of the cylinde and PS perpendicular to OQ. Then QSP and QCO ar similar triangles.

Hence $$QP : QS :: QO : CQ;$$

$$\therefore OQ . QS = CQ . PQ;$$

$$\therefore \frac{PQ}{OQ} = \frac{QS}{CQ} = \frac{PQ + QS}{OQ + CQ}.$$

Now since when x is a very small fraction,

$$\log(1-x) = -x,$$

we shall commit no appreciable error in putting

$$\frac{PQ}{OQ} = -\log\left(1 - \frac{PQ+QS}{OQ+CQ}\right)$$

$$= -\log\frac{OP+CP}{OQ+CQ}$$

$$= \log(OQ+CQ) - \log(OP+CP),$$

which is a form adapted to our method of summation.

The potential of the annulus will now

$$= 2\pi a\rho\,\{\log(OQ+CQ) - \log(OP+CP)\},$$

adding all the elementary differences we have for the potential at O,

$$4\pi a\rho\,\{\log(OB+BC) - \log OC\},$$

and if l be the length of the cylinder, and a its radius, l being assumed great compared to a, this reduces to

$$4\pi a\rho\log\frac{l}{a} = V \text{ suppose},$$

and if Q be the quantity of electricity, $Q = 2\pi a\rho\,.\,l$;

$$\therefore\ V = Q\,.\,\frac{2\log\frac{l}{a}}{l}.$$

If $V=1$, $Q=C$ the capacity;

$$\therefore C = \frac{l}{2\log\frac{l}{a}}.$$

COR. 1. If l be very large compared to a, $\log\frac{l}{a}$ must be very large, and the capacity of the wire is small. We therefore assume that if two pieces of apparatus be connected by a very fine wire, we may neglect the electrification of the

wire, and assume that the charge is simply divided in the ratio of the capacities of the apparatus.

COR. 2. If there be another cylinder outside the one we are considering, separated from it by air, we have for the potential of the Leyden jar so formed,

$$V = \frac{2Q}{l}\left(\log\frac{l}{a} - \log\frac{l}{a'}\right) = \frac{2Q}{l}\log\frac{a'}{a};$$

$$\therefore C = \frac{l}{2\log\frac{a'}{a}},$$

a' being the radius of the outer cylinder.

If the dielectric be different from air and have a specific inductive capacity K,

$$C = K\frac{l}{2\log\frac{a'}{a}},$$

a formula useful in calculating the charge of a marine cable.

EXAMPLES ON CHAPTERS III. AND IV., AND ON GENERAL STATICAL ELECTRICITY.

1. Two particles are charged with quantities q_1 and q_2 of electricity, and another with a quantity $-(q_1+q_2)$, and are placed at the angular points of a triangle. Show that the work done against the two former equals that done by the latter in bringing a + unit up to the centre of the circumscribing circle.

2. Three particles are charged with equal quantities, two + and one − of electricity. Show that the centre of the inscribed circle of the triangle, formed by the three particles, will be on the surface of zero potential if

$$4\sin\frac{\pi - A}{4},\ \sin\frac{\pi + B}{4},\ \sin\frac{\pi + C}{4} = 1,$$

the negatively electrified particle being at A.

3. A rhombus is constructed, two of whose angles are 60°, and a + unit of electricity is placed at each. Two − units are placed at one of the other angles. Show that the potential at the remaining angle is zero.

4. A funnel drawn out into a capillary tube is filled with sulphuric acid, and a gold leaf electroscope having a gold cap is placed underneath it. A rod of sealing-wax, which has been rubbed with gun-cotton, is now held over the funnel, the acid flows out on to the cap of the electroscope and the leaves diverge. Explain the electrical actions which produce the flow of liquid and the divergence of the leaves.

5. An insulated metal lamp is placed in a room in which an electrical machine is at work. The lamp is connected by a wire with a gold leaf electroscope in an adjoining room.

(i) Describe the indications of the electroscope after lighting the lamp and working the machine.

(ii) Describe the indications of the electroscope after the lamp is blown out and the machine stopped.

(iii) An insulated metal cylinder completely encloses the lamp, and is connected with another electroscope. Describe the indications of this electroscope while the machine is in action and the lamp burning, and also after the lamp is blown out.

6. A stick of sealing-wax rubbed with flannel is held over a gold leaf electroscope, and the cap touched.

(i) What will be the state of the leaves?

(ii) If the stick be brought nearer the cap, what will be the indication?

(iii) If the stick is moved further away, what will be the indication?

(iv) What will be the effect of holding a large insulated plate of metal between the sealing-wax and the cap? What effect will the thickness of the plate have?

(v) What will be the effect if the sheet of metal be uninsulated?

(vi) What will be the effect of substituting a plate of paraffin for the metal plate?

7. There are two similar gold leaf electroscopes, one with a point attached to the cap. A piece of sealing-wax rubbed with flannel is held over each of them and removed. Describe the indications of the two electroscopes before and after the removal of the sealing-wax.

8. An insulated metal cylinder, positively electrified, is held with its axis vertical, and a funnel whose nozzle projects along the axis of the cylinder to near its middle has water poured into it.

(i) The funnel is uninsulated, determine the electrical state of the issuing water.

(ii) If the funnel now be insulated, what effect will be produced on the electrification of the issuing jet at first, and after a time?

(iii) In question (i), after the water has run through, the funnel is insulated and removed, what will be the nature of its electrification? Will it differ from that of the funnel in question (ii) after the water is exhausted?

(iv) What effect will be produced on the issuing jet, by connecting the funnel with the cylinder?

(v) In (i) the issuing stream of water flows into another funnel, which is contained inside a second insulated cylinder and connected with it. What will now be the state of the issuing stream, and what would be the electrical state of the second cylinder supposed neutral at first?

(vi) Will the potential of the lower cylinder go on increasing without limit; or if there be a limit, on what will it depend?

(vii) Show how an arrangement depending on the principle of the preceding questions could be constructed, by which a small charge given to a Leyden jar could be augmented to a high degree.

9. A positively electrified particle repels every other positively electrified particle, but two conductors charged with positive electricity do not necessarily repel each other. Explain this apparent paradox.

10. Show that two equal conductors similarly placed with respect to each other, both raised to the same potential, and insulated, always repel each other.

11. Show that if the potentials of the two conductors in the last question differ ever so little, they will, at great distances, repel each other, but at very near distances (supposing no spark to pass) they will attract each other.

12. Two very thin parallel plates are pressed closely together, electrified and insulated. Show that the work done by them during separation equals half the whole energy of the electrification.

13. Two thin parallel plates are electrified to the same potential, draw a rough diagram of the lines of force.

14. The two thin plates in the preceding question are electrified to different potentials. Draw the lines of force, and show on what principle we determine whether there will be attraction or repulsion between the plates. (See Art. 65, Cor. and Art. 87.)

15. Two spheres of radii 4 and 5 centimetres, are connected by a long and fine wire, find the proportion in which a charge communicated to the system is divided between the spheres.

16. A sphere of radius one decimetre, is connected by a long wire with a plate one decimetre square, which has at distance one millimetre from it another parallel plate connected with the earth. Find the ratio in which a charge will be divided between the plate and sphere. Calculate also the numerical capacity of the whole system.

Ans. π to 25; $\dfrac{10(\pi+25)}{\pi}$.

17. A plate whose radius is one decimetre, is charged with a unit of electricity, and moved till distant one millimetre from a similar plate connected with the earth. Compare the potential of the plate before and after the movement of the plate. *Ans.* 25π to 2.

18. Two spheres, each one decimetre in radius, are connected by a wire. A third conducting sphere is concentric with and envelopes one of the spheres, and is also connected with the earth: the distance between the surfaces being two millimetres. Show in what proportion a charge communicated to the system is divided. *Ans.* 51 to 1.

19. A Leyden jar one millimetre thick, and having 1 sq. decimetre surface, is fully charged by 5 turns of an electrical machine. How many turns are necessary to charge a battery of 40 square decimetres, and 6 millimetres thick? *Ans.* $33\frac{1}{3}$.

20. With same data as ques. 19, what fraction of full charge will be communicated to a battery of 20 sq. decimetres, ·5 mil. thick, by 45 turns of the machine? *Ans.* $\frac{9}{40}$.

21. With same data as ques. 19, a battery having 200 sq. decimetres is charged by 500 turns of the machine. Find the thickness. *Ans.* 2 m.m.

22. Compare the energy of discharge of two batteries, one of 20 sq. decimetres, and the other of 80 sq. decimetres, both of same thickness, and charged to same potential.

23. Compare in last question the energy of discharge of the two batteries, supposing one charged with 80, and the other with 150 turns of the machine, neither being supposed fully charged. *Ans.* 256 to 225.

24. Compare the energy of discharge in two batteries, one of 15 sq. decimetres and the other of 60 sq. decimetres, each charged by the same number of turns of the machine, the thickness being the same in both. *Ans.* 4 to 1.

25. Compare the capacities of two batteries, one of 40 sq. decimetres, 1 mil. thick, the other 100 sq. decimetres, 1·5 mil. thick. *Ans.* 3 to 5.

26. Compare the potentials of two batteries, one of 30 sq. decimetres surface, $1\frac{1}{2}$ mil. thick, the other of 80 sq. decimetres surface, ·8 mil. thick, charged with equal amounts of electricity. *Ans.* 5 to 1.

27. Compare the potentials of the two batteries in the last question, supposing one charged with 10 turns of the machine, and the other with 40 turns, supposing neither fully charged. *Ans.* 5 to 4.

28. Compare the amounts of heat evolved in the discharge of the two batteries of the last question. *Ans.* 5 to 16.

29. A battery of 20 sq. decimetres charged with 40 turns of the electrical machine, will just puncture glass ·3 mil. thick. What extent of coated surface of the same thickness, charged to the same potential, will pierce a sheet of glass 3 mil. thick?

30. Two parallel conducting plates are connected, one with the earth, and the other with a source of electricity of constant potential. A positively electrified particle falls from the positive to the negative plate. Show that:

(i) The acceleration on the particle varies inversely as the distance between the plates.

(ii) The time of falling is directly proportional to the distance between the plates.

(iii) The velocity acquired in falling is independent of the distance.

31. A gold leaf electroscope is connected by a long wire with various points in succession on an electrified conductor, the distribution being (1) free, (2) induced. What difference (if any) will there be in its indications?

32. In what respects will the indications of the preceding question differ (1) from those obtained by touching the various points with a proof plane and bringing it near the electroscope, (2) from the results obtained by suspending pith balls at various points on the conductor?

33. The plates of a condensing electroscope are connected by a long fine wire, electrified and separated. Will there be any change observed in the divergence of the leaves during separation?

34. How are the potentials of the surfaces of a charged

and insulated Leyden jar affected by letting down into it a conductor—

(i) Connected with the earth?

(ii) Completely insulated?

(iii) Completely insulated, but left with one half extending outside the jar?

35. Two plates, having gold leaves attached to their faces, are charged as a Leyden jar, and insulated. The distance between the plates is now varied. Discuss fully the changes in the state of the gold leaves as the distance is varied.

36. A Leyden jar is charged and placed on the cap of a gold leaf electroscope. A small body, neutral or electrified, is brought near the knob of the jar and then removed. Describe and explain all the indications of the electroscope (1) when the body is neutral, (2) when positive, (3) when negative.

37. What differences would there be in the preceding question, if the body be allowed to touch the knob of the jar?

38. A series of n jars, whose capacities are $C_1, C_2, C_3, \ldots$ are charged by cascade and fitted up as a battery. Show that if C be the capacity of the battery, and we neglect the free charges,

$$\frac{n}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots\ldots\ldots$$

39. Hence show that the capacity of a battery charged by cascade is greater than that of the jar of lowest and less than that of the jar of highest capacity.

40. Three jars are connected by fine wires and charged by cascade. Show how to calculate the electrification of the system, making allowance for the free charge.

Ans. If C_1, C_2, C_3 be the capacities for bound charge,

i_1, i_2, i_3 be free capacities of charges on inner coats,

o_1, o_2, o_3 be free capacities of charges on outer coats,

and V be potential of source, the quantities are respectively

$$\frac{(C_2+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}{(C_2+C_1+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}\cdot C_1V,$$

$$\frac{C_1(C_3+i_3+o_2)}{(C_2+C_1+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}\cdot C_2V,$$

$$\frac{C_1C_2C_3}{(C_2+C_1+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}\cdot C_3V.$$

41. A number of conductors of capacities C_1, C_2,... are raised to potentials V_1, V_2,... respectively, and afterwards connected by fine wires so as to form one conductor. Show that the potential of the conductor is given by

$$\frac{\Sigma CV}{\Sigma C}.$$

42. Show that in the preceding system the energy of the whole conductor is to the energy of the separate conductors as $\{\Sigma(CV)\}^2$ is to $\Sigma C\times\Sigma\,(CV^2)$.

43. Two equal jars are charged one positively and one negatively to the same potentials. The inner and outer coats are then connected by wires. What will be the state of each jar?

44. In the preceding question, does electricity pass from one outer coat to the other when both are at zero potential?

45. In the last question but one, what would happen if the outer coats were insulated and the inner coats connected by a wire?

46. Two spheres, whose radii are r_1 and r_2, are in regions of potentials V_1 and V_2 and are connected by a fine wire.

(i) Find the potential of the system.

(ii) Find the free charges, supposing the wire to be suddenly cut by a pair of scissors with glass handles.

47. A ball is insulated and held within a Leyden jar, being connected by a wire with an electroscope outside.

(i) What will be the indication of the electroscope when the jar is first charged?

(ii) What would be the simultaneous indication of another electroscope entirely within the Leyden jar and connected with the ball?

(iii) If after charging the jar the ball be touched by the finger, what will be the indication of the outside electroscope?

(iv) How will the indication of the electroscope be affected if the jar gradually leak?

(v) If the Leyden jar be charged and insulated before the introduction of the ball, how will the introduction of the ball affect an electroscope connected with its outer surface?

48. A plate of radius a has another plate of the same radius at a small distance T from it.

(i) If the system be charged as a Leyden jar, compare the free and bound charges. *Ans.* As $4T$ to $a\pi$.

(ii) If the plates of the Leyden jar be insulated and removed to a distance t, find the potential of the plates and the amount of the free and bound charges. Discuss both the cases in which t is greater and less than T.

(iii) The plate receives a charge Q of electricity and is moved till it is distant T from the second plate, which is connected with the earth. Find the potential and the amounts of the free and bound charges.

Ans. Potential $= \frac{\pi}{a} \cdot \frac{4T}{4T + a\pi} \cdot Q.$

(iv) If $a = 1$ decimetre and $T = \cdot 01$ cm. and the system be charged as a Leyden jar, calculate the rise in potential of each plate when the plates are entirely separated. How is this applied in the condensing electroscope?

Ans. Ratio of 1 to 250π.

(v) The two plates are charged as a Leyden jar and the positive plate is removed, the negative plate being left uninsulated. Find the whole work done. (See Art. 99.)

49. A Leyden jar is charged in the usual manner and insulated. The knob is now touched by the finger. Find the change in potential of the two coats and calculate the energy of the discharge.

50. Two Leyden jars charged to different potentials have their knobs brought for an instant into contact. Calculate the energy of the spark which passes.

51. A Leyden jar is charged, and the charge divided with another equal Leyden jar, which is uncharged. Show that one-half the whole energy of the system runs down in the spark.

52. Find the energy expended in charging a conductor of known capacity to a given potential by means of a unit jar, whose potential of discharge is known. Show that it will be independent of the capacity of the unit jar.

53. A Leyden jar is charged and fitted up with $(n-1)$ uncharged similar jars to form a battery, show that the whole energy is only $\frac{1}{n}$ of the energy of the single jar.

54. A plate is placed between two equal and parallel plates, and the three are electrified to given potentials, the middle plate being highest. Find the position of equilibrium of the middle plate.

55. If the middle plate, when in a position of equilibrium, be removed, find the amount of its charge.

56. Show that the equilibrium in the preceding example is unstable.

57. What is the nature of the equilibrium of the needle in a quadrant electrometer?

58. Explain the experiment of 'Mahomed's Coffin.' A small chip of gold leaf, pointed at one end and blunt at the other, is thrown into the air near the knob of a charged Leyden jar, and is observed to remain freely suspended for some time.

59. Show how to find the potential at any point between two parallel plates electrified to different potentials.

60. A small sphere is insulated and placed between the parallel plates in the last question, show how to determine its potential.

61. If a small insulated sphere be placed between two concentric spheres, charged as a Leyden jar, show how to find its potential.

62. Show that any symmetrical conductor, placed symmetrically in a uniform field of force, will have the same potential as that at its centre, supposing the conductor removed.

63. A sphere of radius r is insulated in a large uncharged Leyden jar without contact with the walls. It is connected by a long wire with another sphere of radius R, insulated in a region of zero potential. The jar is charged, and its potential rises uniformly at the rate of v units per second; find the rate of flow of electricity through the wire.

Ans. $\dfrac{rR}{r+R} v$ units per $1''$.

64. Deduce the rate of flow in the preceding example, supposing the wire to have its distant end to earth.

65. A soap-bubble is blown and afterwards electrified. Find an expression for the radius of the soap-bubble that the internal pressure on the soap-film may be constant as the electrification proceeds.

Ans. $\rho^2 = \dfrac{\Pi}{2\pi}\left(1 - \dfrac{a^3}{r^3}\right)$, where Π is the constant pressure, a the initial radius, r, ρ the radius and electrical density at any time during electrification.

66. In *Holtz's Machine* or in the *electrophorus,* any amount of electricity however small is made to produce an amount of electrical separation as great as we please. Can this be reconciled with the principle of conservation of energy?

67. An unelectrified conductor at zero potential on being insulated and introduced into a space at potential V, assumes the potential of the space. Show how this can be reconciled with the principle of conservation of energy.

68. Point out how the same principle is satisfied in the water-dropping apparatus described in ques. 8.

69. Show that in a system of equipotential surfaces round an electrified sphere, the distances of the consecutive members of the system of equipotential surfaces from the centre of the sphere form an Harmonical Progression.

70. If any system of equipotential surfaces be freely electrified, the capacity of any surface varies inversely as its potential.

71. Show that the rate of movement of any equipotential surface as the electrification proceeds at a uniform rate, varies inversely as the product of the force at the point on the surface multiplied by the capacity of the surface for a free electrification.

72. Show that in the case of an electrified sphere, the rate of electrification is equal to the velocity of any equipotential surface multiplied by its potential.

73. If a sphere be at zero potential, and have its centre at a distance f (> radius) from a particle having m units of electricity, show that the quantity of electricity on the sphere is $-\frac{am}{f}$, a being the radius.

74. If the sphere be charged with Q units of electricity, and brought near a point having m units of electricity, find the potential within the sphere.

Ans. $\frac{Q}{a}+\frac{m}{f}$.

75. If a hollow sphere be charged with Q units of electricity, and have a particle charged with $-q$ units introduced through a small aperture, find the position of the particle that the potential at the centre may be zero.

Ans. Distance from centre $=\frac{aq}{Q}$.

76. A sphere near an electric system is brought to zero potential and insulated. On being removed the potential of the sphere is found to be $-V$. Show that the sphere occupied such a position that the potential at its centre due to the given system was $+V$.

77. Two spheres of unequal radii are charged to the same potential, insulated, and brought near to each other till a spark passes. Find in which direction the spark will pass between the spheres.

78. To find the work done in moving a particle charged with a given quantity of electricity, from any given point within a sphere to its centre.

Let P be the position of the particle charged with m units of electricity and CPT a diameter, T being conjugate to P. Let P', T' be a pair of conjugate points on the same diameter near to P, T.

The force on m at P is only that due to attraction of $-\frac{a}{f}m$ at T;

$$\therefore \text{ Force} = \frac{m \cdot \frac{a}{f}m}{PT^2} = \frac{am^2f}{(a^2-f^2)^2}.$$

But if PQ, $P'Q'$ be drawn perpendicular to CPT,

$$a^2 - f^2 = CQ^2 - CP^2 = PQ^2;$$

$$\therefore \text{ Force} = \frac{am^2 \cdot CP}{PQ^4}.$$

Hence average force over $PP' = \frac{am^2}{2} \cdot \frac{CP + CP'}{PQ^2 \cdot P'Q'^2}$;

$$\therefore \text{ Work done through } PP' = \frac{am^2}{2} \cdot \frac{(CP + CP')(CP - CP')}{PQ^2 \times P'Q'^2}$$

$$= \frac{am^2}{2} \cdot \frac{CP^2 - CP'^2}{PQ^2 \cdot P'Q'^2} = \frac{am^2}{2} \cdot \frac{P'Q'^2 - PQ^2}{PQ^2 \cdot P'Q'^2}$$

$$= \frac{am^2}{2}\left(\frac{1}{PQ^2} - \frac{1}{P'Q'^2}\right).$$

Adding the whole work from a given point K to the centre

$$= \frac{am^2}{2}\left(\frac{1}{CA^2 - CK^2} - \frac{1}{CA^2}\right) = \frac{f^2m^2}{2a(a^2 - f^2)},$$

supposing $CK = f$.

79. A sphere is at zero potential, find the work done in removing a particle charged with a given quantity of electricity from any external point to an infinite distance.

80. A sphere is charged with a given quantity of electricity, find the work done in moving a particle from any given external point to an infinite distance.

81. A very large insulated circular plate has a particle charged with m units of electricity very near to its centre, find the potential of the plate.

82. In the last question, find the position of the line of neutral electrification.

Ans. $r^6 - a^2p^2r^2 = a^2p^2(a^2 + p^2)$ where a = rad. of plate, p = distance of particle from centre and r the distance of the neutral line from the particle.

83. Show that in the case of an uncharged sphere, under the influence of an electrified particle, all points in the neutral line and the centre of the sphere are at the same distance from the electrified particle.

84. A uniformly electrified ring is placed in a diametral plane of a sphere at zero potential and is concentric with it, find the density of the charge at either pole.

85. If a ring, having the same radius as a sphere, be placed in a tangent plane to the sphere, so that the point of contact is the centre of the ring, compare the electrical density at the centre of the ring and at the opposite extremity of the diameter, the potential of the sphere being zero.

86. Given the amount of electrification of the ring, find the amount of the whole induced charge in each of the two last questions.

87. If the ring be placed in a symmetrical manner inside the sphere, find the density at the two poles.

88. A closed region, whose surface is a bad conductor, encloses a very delicate electrometer; electrified bodies are moving about with great velocity outside the closed region, will the electrometer give any indication?

89. What would be the best form of electrometer for conducting the above experiment, and in what manner would you fit it up to make the indications as great as possible?

90. If the movement were one of rotation round the closed space, so as to keep the moving bodies on the whole at a constant distance, would there be any indications? How would an observer, placed outside the region, proceed to make observations in this case?

91. If you were in a closed space, having only a small aperture, how would you proceed to determine the electrification of the space?

92. How far does the method you employ in the preceding question apply to determine the *absolute* electrification of the earth?

93. What would be the electrical state of a sky-rocket just before reaching the earth?

94. A balloon is allowed to ascend from the earth carrying a burning match, which is kept connected with one terminal of a quadrant electrometer by means of a fine insulated wire, which is let out as the balloon ascends. During the first hundred yards the potential rises gradually at the rate of 1^0 per 10 yards of ascent. After this the register is constant for 20 yards, for the next 50 yards it falls at the rate of 1^0 per 15 yards, and again rises uniformly at the rate of 1^0 per 12 yards of ascent. What inferences as to atmospheric electricity would be drawn from these observations?

95. Explain why in a Leyden jar the loss of charge appears more rapid a few minutes after first charging than it does afterwards.

96. A Leyden jar is charged and left for a few minutes, when its charge is divided by instantaneous contact with another equal jar. State what will be the condition of the two jars a few minutes afterwards.

97. A Leyden jar made of a plate of shellac, coated on both sides, is charged, discharged and the coats removed. What will be the electric state of the surface of the shellac, and how will it vary with time?

98. If two spheres, placed in oil of turpentine, be charged to given potentials, will the force between them be greater or less than in air?

99. If two spheres be charged with given quantities of electricity and placed in oil of turpentine, will the force between them be greater or less than in air?

100. A metal sheet is placed between two plates of non-conducting matter, whose inductive capacities are K and K', and their thicknesses t and t', and two other metal sheets are placed outside the plates. The inner sheet is kept at potential V, while the outer sheets are at zero. Compare the charges on the outer sheets on being insulated and removed.

101. Faraday constructed a room coated externally with tinfoil and furnished with an aperture or window. The whole room was insulated on glass legs, and could be powerfully charged by a large frictional machine.

(i) On charging the room, what effect would be produced on electrometers placed inside it?

(ii) How would a person inside proceed to determine the external electrification of the room?

(iii) If a frictional machine is carried inside the room and worked, the rubber being connected with the walls of the room, how will a gold leaf electroscope, placed outside in contact with the external surface, be affected?

(iv) If a ball be charged inside the room, insulated and carried out, what effect will be produced on the electroscope?

(v) A number of conductors are charged from the machine within the room, and suspended by silk threads within the room, how will these affect the external electroscope?

102. A ball is electrified and held above a metal plate, which is then touched by the finger, what indications would be obtained by testing the plate at various points above and below with a proof plane?

103. A metallic ball is lifted by a silk fibre on to the top of a rod of sealing-wax, the lower part of which has been rubbed with a silk handkerchief, what indications would be obtained by touching it at various points with a proof plane?

104. What differences would there be in the last question if the ball had been placed on the sealing-wax by hand?

105. Two large spaces are constructed, which are kept at constant potential, one A at potential V_1, the other B at potential V_2, supposing $V_1 > V_2$. Two spheres of equal radii are placed in these regions insulated from them, and connected by a fine wire also insulated.

(i) What will be the potential and the amount of charge on each sphere?

(ii) What would be the indication of an electroscope placed in space A, and connected with its sphere?

(iii) What would be the indication of an electroscope placed in space B, and connected with its sphere?

(iv) A burning metal lamp is placed on the sphere in region A, how will the indications of the two electroscopes be affected?

(v) If a burning metal lamp be placed on each sphere, how will the indications be affected?

(vi) What will be the effect on the indications of the electroscopes if the wire be at some point to earth?

106. A sphere of radius one centimetre is charged with a unit of electricity and placed in a space at potential 10, what will be the potential of the sphere?

107. A sphere of radius unity is introduced into a place at potential 5, and then connected with the earth. What will be its free charge on being insulated and removed?

108. A conductor whose capacity is 4, is introduced into a room whose potential is 4, and the conductor is then brought to potential 3, insulated, and removed. What will be the amount of the electrification?

109. A conductor whose capacity is 6, is charged with 12 units of $-$ electricity, and placed in a region at potential 3. What will be the potential of the conductor?

110. A conductor at zero potential is in a space at potential 8; on being insulated and removed it has 24 units of $-$ electricity. What is its capacity?

111. A conductor of capacity C is charged with Q units of electricity, and put in a space at potential V. What will be the potential of the conductor?

112. A conductor is brought to zero potential in a space at potential V. On being insulated and removed it is found to have $-Q$ units of electricity. What is its capacity?

113. A conductor of capacity C is placed in a region at potential V, and brought to potential V'. Find its charge.

114. If the prime and negative conductors of an electrical machine have equal capacities, show that the effective working of the machine is at first diminished by one half, when the negative conductor is insulated.

115. If the capacities in the last question are in the ratio C_1 to C_2, find the ratio in which the effective working is at first diminished by insulating the negative conductor.

116. If an electrical machine be placed in the open air at a height h from the earth, and worked (with rubber uninsulated) till the prime conductor has a charge e of electricity, when the earth connection is broken; show that negative electricity is spread over the earth with a density at any point represented by $\frac{he}{2\pi r^3}$, where r is the distance of the point from the machine.

117. Show also that the change produced in the potential of the earth is to the potential of the conductor as $-hC$ to R^2, where C is the capacity of the conductor and R the radius of the earth.

118. Show that if the machine be worked in a closed room there is absolutely no change in the potential of the earth.

119. How far do the results of the preceding questions affect our taking the earth as our zero point, and do they point to an advantage from working in the open air or in a room?

CHAPTER V.

THEORY OF THE VOLTAIC CELL.

114. We have stated that when two conductors brought by means of an electrical machine to different potentials are joined by a conducting bridge, an equalization of potential takes place through the bridge, which we may represent as a flow of electricity from the place of higher to that of lower potential, or briefly as a current of electricity. We have, moreover, calculated the mechanical equivalent of such a discharge, the energy being converted into heat in the bridge, or into work external to the bridge in a variety of ways. The phenomena belonging to the bridge while the current is passing form the special subject for consideration in Voltaic Electricity or Galvanism.

The current obtained by means of the common form of machine is a single instantaneous discharge, or a rapid succession of such instantaneous discharges, and therefore ill adapted for the production of the class of phenomena to which we have alluded. They can be observed to perfection by means of the galvanic battery, in which the electrical separation takes place with such rapidity, that the successive discharges, if they exist, cannot be separated by the most delicate tests. We must bear in mind, however, that the differences of potential with which we are concerned are extremely minute compared with those obtained in the machine, while the quantity of electricity in motion is incomparably greater. To return to our old hydrostatical analogy, the machine current is a tiny stream tumbling down a precipitous hill-side, the galvanic current is a vast lake flowing through an almost level valley.

115. Before proceeding to the phenomena themselves, we shall consider the connection between the two modes of generating electricity.

In all forms of electrical machine the source of electricity is ultimately the friction of two bodies of different substances, which, when rubbed together, appear to exercise an unequal attraction for the opposite electricities, which were at first neutral in both bodies. The result of this unequal attraction is the production of a difference of potential between the bodies, this difference, while they are in contact, depending on the nature of the rubbing surfaces, and on the amount of rubbing.

The energy represented by this difference of potential is derived from the mechanical rubbing, as also are the heat and change in character of the two surfaces which accompany it.

For the development of the galvanic current, it appears necessary that there should be at least three heterogeneous bodies arranged in a circuit, one of such bodies, at least, capable under some conditions of being decomposed and forming a chemical compound with one of the other two.

116. Suppose A, B, C to be three such bodies, of which A, B exercise a chemical affinity for each other; the development of the current has been attributed to one of two causes:—

(i). To the differences of potential produced at the three places of contact, A with B, B with C, and C with A. This is Volta's or the Contact Theory.

(ii). To the chemical attraction between A and B, which throws the circuit into a state of polarization; the resulting chemical action being accompanied by an electrical discharge round the circuit; the galvanic current being the result of a rapid succession of such alternate polarizations and discharges. This is Faraday's or the Chemical Theory.

117. These two theories of the action of the cell have been warmly debated among physicists, our countrymen for the most part siding with the more recent theory of Faraday, while continental physicists have for the most part accepted the older theory of Volta, though somewhat modified. The point of dispute amounts briefly to this: Volta

recognized that one of the three substances in the circuit must be a fluid; Faraday, however, seeing that the chemical composition of this fluid was, in all cases, altered by the passage of the current, attributed the current solely to this chemical action. As a crucial experiment he constructed a cell in which were two metals and one fluid, the fluid being (for fluids) a good conductor, but not capable of acting chemically on either of the metals. He showed, by the most delicate tests, that in this case there was no current in the cell. This, in his opinion, entirely overthrew Volta's Theory. More recently, however, the perception of the law of Conservation of Energy, first put forth by Helmholtz, has shown that in the crucial experiment relied on by Faraday the existence of a current would have been an independent creation of energy. This has again opened the question, and experimenters have diligently set themselves to work to put the theory of Volta again to the test of exact experiment.

So great, however, is the intrinsic difficulty of these experiments, that it is hardly too much to say that at present in no single instance has a difference of potential been directly shown between two bodies, independent of the gaseous medium between them, of the pressure with which they are brought together, and of the friction with which they are separated; the existence of such a difference of potential in every case lying at the very foundation of Volta's theory.

118. Not only is experiment on this subject very difficult, but the theories deduced from it are highly speculative. In our supreme ignorance as to what is going on in the ultimate molecules of a body, it would be rash in the extreme to make more than very tentative hypotheses on either their mechanical or electrical properties.

The hypotheses in this chapter are made in this tentative way, with the hope that they may enable the student mentally to picture the actions with the results of which we assume him to be cognizant.

119. We have no intention of entering into experimental details, beyond the simplest citation of the results on which the theory we set forth is founded. For more full accounts of the work of experimenters, we must refer the

student to the first chapter of Wiedermann's *Galvanismus*, and to the very full references to original memoirs he has given.

120. The experiments may be briefly classified as follows:

(i). Electrical excitement between two metals.

(ii). Electrical excitement between a metal and a fluid.

(iii). Electrical excitement between two fluids.

(iv). Electrical excitement between one or two metals and one or two fluids—the voltaic cell.

(v). Electrical excitement by gases condensed on metals—polarization.

121. I. *Electrical excitement between two metals.* These experiments were originally conducted by Volta with the help of his pile and condensing electroscope, and on them he was led to found the theory associated with his name. The objection to them lies in the fact that there are in most cases other contacts, besides the one being investigated, which may have contributed to the observed effect.

The best method of conducting the experiment is by means of an electroscope whose two condensing plates are of zinc and copper. A wire, either of zinc or copper, is then bent once so as to come in contact at one end with the copper and at the other with the zinc plate. On removing the wire and lifting the upper plate the leaves diverge, showing that zinc carries off at contact positive, and copper negative electricity. The great difficulty in all these experiments lies in proving that the clean metal surfaces themselves come into actual contact, first, because both the oxygen and damp in the air may act chemically on the surfaces; and secondly, on account of the remarkable property possessed by metals of condensing gases on their surfaces. Nor are these objections entirely overcome when experiments are conducted in gases which do not act on the metals, in air carefully dried, or even in a vacuum, since the condensed gases adhere with pressure greater than that of an atmosphere. They appear, however, to lose much of their force when the metals are soldered together at their junction. Sir William Thomson has applied with metals soldered at their junction a

method identical in principle with his Quadrant Electrometer. On suspending over the compound bar (shaped in the form of a circular annulus), a counterpoised aluminium needle which, when unelectrified, points towards the junction, he found that on electrifying the needle to a high potential it deviated towards one metal or the other, showing a difference of potential between them. Other methods of experimenting on the contact theory by the same author may be found in his *Papers on Electrostatics and Magnetism,* Chaps. XXII. and XXIII.

122. Physicists starting with Volta have arranged the metals in 'tension series,' that is to say in an order such that any one in the list becomes positive when in contact with any one that succeeds, but negative by contact with any one that precedes it. We subjoin those given by Volta and Seebeck.

Volta's series: zinc, lead, tin, iron, copper, silver, gold, carbon, graphite, manganese.

Seebeck's series: zinc, polished lead, tin, raw lead, antimony, bismuth, iron, copper, platinum, silver.

123. In quantitative relation we will represent the difference of potential between two substances iu contact by a vertical stroke placed between their chemical formulæ, making the convention that that which is to the right of the line is at the higher potential. Thus Cu | Zn represents the difference of potential between zinc and copper in contact, assuming the zinc at higher potential than the copper, while the symbol Zn | Cu may be taken as equal symbolically to − Cu | Zn.

124. Volta, by means of a straw electroscope, expressed in terms of an arbitrary unit some of these differences: thus he gives

$$\begin{array}{ll} \mathrm{Pb} \mid \mathrm{Zn} = 5, & \mathrm{Sn} \mid \mathrm{Pb} = 1, \\ \mathrm{Fe} \mid \mathrm{Sn} = 3, & \mathrm{Cu} \mid \mathrm{Fe} = 2, \\ \mathrm{Ag} \mid \mathrm{Cu} = 1, & \mathrm{Ag} \mid \mathrm{Zn} = 12, \\ \mathrm{Cu} \mid \mathrm{Sn} = 5, & \mathrm{Cu} \mid \mathrm{Zu} = 9. \end{array}$$

Now in this table if we compare, for instance, Ag | Zn with that of the preceding pairs we have

$$\begin{aligned} &Ag \mid Cu + Cu \mid Fe + Fe \mid Sn + Sn \mid Pb + Pb \mid Zn \\ &= 1 \quad + \quad 2 \quad + \quad 3 \quad + \quad 1 \quad + \quad 5 \\ &= 12 = Ag \mid Zn. \end{aligned}$$

Similarly we have

$$Cu \mid Sn = 5 = Cu \mid Fe + Fe \mid Sn,$$

and $Cu \mid Zn = 9 = Cu \mid Fe + Fe \mid Sn + Sn \mid Pb + Pb \mid Zn.$

Now, although the numbers themselves are but the roughest possible approximations, both experiment and theory lead us to the conclusion that the fact expressed by the above formulæ is true, namely, that of all the substances belonging to Volta's tension series, the difference of potential between the extreme ends of a compound bar, composed of any number of these substances arranged in series, will be the same as if the two extreme substances were joined directly.

125. To explain how the contact of two substances may produce a difference of electrical potential, we must make some assumptions as to the properties of the molecules in the bodies. This is not a place to discuss the evidence on which the molecular theory is based, but the student will find an admirable exposition of an elementary character in the last chapter of Prof. Clerk Maxwell's *Theory of Heat*, or a more elaborate treatment in the writings of Prof. Clausius. We must make the following assumptions.

(i). That in liquids and gases the molecules of which they are composed are in a constant state of rapid motion, the molecules of each kind of gas having a definite mass but a great variety of velocities.

(ii). That the molecules of the gas or liquid are constantly impinging on each other and bounding off, like billiard-balls, without loss of energy.

(iii). That the envelope enclosing a gas is subject to a constant bombardment by the molecules: the pressure of the gas upon the envelope being measured by their momentum.

(iv). That in a perfect liquid, though the average velocity of the molecules is much less than in a gas, there is

the same freedom of movement; but in all known liquids there exists a constraining force limiting their free motion, as shown by their viscosity and capillary attraction.

(v). That in solids the constraining force on the molecules is very much greater than in liquids, allowing each molecule but a very short amplitude of excursion; in two different solids the masses of the molecules, the average amplitudes and periods of excursion, being different.

From the last statement it can be easily seen that when the molecules of two different solids impinge on each other, as at the surface of contact, they cannot accommodate themselves to each other's motion, but constrain each other, this constraint producing a loss of energy. If, however, the two solids are of the same kind and at the same temperature, the molecules on each side of the surface of contact are swinging in exactly the same manner, and can easily accommodate themselves to each other's motion without more constraint than exists in the solid part of either body. It is this loss of energy owing to the unsymmetrical swinging of the molecules at the surface of contact which reappears as difference of potential between the two solids, or as the energy of electrical separation.

The opposed electricities so separated will, for the most part, be heaped up on either side of the plane of separation by a Leyden jar action.

Let A be the area in contact in any particular instance,
Q the quantity of electricity separated,
V the difference of potential produced.

Then the energy of electrical separation is (Art. 77) $\frac{1}{2}QV$.

The molecular energy abstracted will be proportional to the area in contact, and may be written mA, where m is a constant depending on the nature of the two surfaces.

Hence $$mA = \tfrac{1}{2}QV.$$

Again, in a Leyden jar of given substance and thickness, the quantity of the accumulation is proportional jointly to the difference of potential and to the area of the surface of the jar. Hence we may write

$$Q = nA \,.\, V,$$

where n is another constant, depending only on the nature of the two bodies. Hence we have

$$mA = \tfrac{1}{2}\, nA \, . V^2,$$

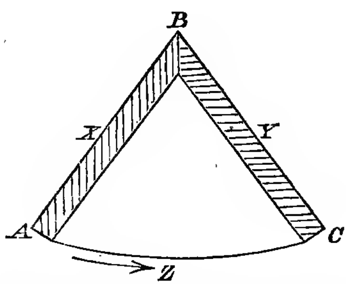
$$\text{or } V^2 = \frac{2m}{n}.$$

Hence V or the difference of potential produced by contact is independent of the shape of the bodies and of the area in contact, depending only on the substances concerned.

We have neglected here the small portion of the electrification which will distribute itself over the two bodies according to electrostatic laws, maintaining the two bodies at a constant potential throughout their mass. This will in all cases be exceedingly small, corresponding to the free charge in a Leyden jar. Since for two given bodies $Q = nA \, . V$, and V has been shown to be constant, the quantity of electricity separated is proportional to the area. To make the electroscope indication after separation as great as possible, it would be best to grind together a zinc and copper plate as for the condensing plates of an electroscope. If two such plates be separated by insulating handles, the charge carried away can be made evident by an electroscope only moderately sensitive.

126. Suppose now two bodies AB, BC to be joined at one end B. In virtue of the contact one of them, suppose

Fig 40.



AB, acquires a higher potential than the other, BC. If now we could join A and C by a body which behaved *only* as a

conductor. a flow of electricity would take place between A and C tending to equalize their potentials; the contact at B would develope a fresh difference of potential, and we should have a continuous current through AC. This current would be a source of energy, and we should, in this case, have an unfailing source of energy. The law of conservation of energy shows this to be impossible. Thus we learn that the contact of C, with A on one side and with B on the other, must produce differences of potential whose aggregate effect is to counteract the difference at the junction B. Or calling X, Y, Z the three bodies, according to our notation

$$Y \mid X = Y \mid Z + Z \mid X,$$

$$\text{or } Y \mid Z + Z \mid X + X \mid Y = 0.$$

This must be regarded as a fundamental relation in the case of all bodies whose molecular condition remains unaltered. We have already (Art. 124) cited it as an experimental property of the bodies comprising Volta's tension series.

127. II. *Electrical action between one metal and one fluid.*

Again, it is difficult to arrange that the only contact of heterogeneous bodies shall be the metal and fluid it is desired to experiment upon. The following method seems however free from exception. A metal plate (suppose of zinc) is screwed on to a very sensitive electroscope, and on its upper surface a very thin sheet of glass is laid, the glass having been carefully coated on its under surface and edges with shellac varnish. On this glass plate the fluid (suppose water) must be spread by a brush or used to saturate a sheet of blotting-paper which is laid on the glass. A zinc rod is now brought in contact with the metal at one end, and laid in the fluid at the other. On removing the zinc rod, and lifting up, without direct contact, the glass plate, the leaves of the electroscope are found to diverge with negative electricity. Since in this experiment the zinc wire remains wet after removal, it carries with it the electricities bound across the place of contact, and the phenomenon observed is due only to the small free charge distributed over the zinc plate, assisted by induction across the air and glass plate. The result is consequently extremely feeble.

By other modes of experimenting, various precautions are taken to eliminate as far as possible the effects of contacts between foreign bodies. We must however content ourselves with quoting the chief results arrived at by Becquerel and Péclet. They find that as a general rule the metals are negative to liquids, manganese forming a remarkable exception. Silver, gold, and platinum are also weakly positive by contact with most acids.

They also notice that the intensity of electrical separation and of chemical action between an acid and a metal frequently correspond, but that this is by no means invariably the case. It appears also that the fluids cannot be placed with the metals in Volta's tension series. For in the case given above water, being positive to zinc, will be placed before zinc in the tension order. We should therefore expect the electrical excitement of the water with the relatively high zinc to be much weaker than with the relatively low copper—exactly the reverse being the fact.

128. We conclude that conductors must be arranged in two classes.

I. Those which follow Volta's tension law, comprising the metals, and some compound bodies. For these we may assume the fundamental relation between any three X, Y, Z,

$$X \mid Y + Y \mid Z + Z \mid X = 0.$$

II. Those which do not follow Volta's tension law, comprising most compound fluids—water, acids, solutions of salts, and some other conductors (including possibly glass and shellac)—which are decomposed by the passage of the current.

If one or more of the three conductors X, Y, Z belong to this class, we may not assume

$$X \mid Y + Y \mid Z + Z \mid X = 0.$$

129. III. *Electrical excitement between two fluids.*

Kohlrausch has shown the electrical excitement between two liquids in the following manner. He suspended two plates of glass horizontally by silk threads, placing on the upper

surface of one plate, and on the lower surface of the other, sheets of blotting-paper moistened with the solutions under consideration (nitric acid and potassium carbonate suppose). He then lowered down the latter of the two plates until it was distant but a very small space from the former, the two sheets of blotting-paper all but touching each other. These sheets of blotting-paper he connected by insulated threads moistened with various liquids, or by metallic wires, testing the charge after each experiment by a sensitive electroscope. He showed by this means that there existed an excitement owing to the contact of liquids only, but that this excitement was very small compared with that between liquids and metals.

Other methods depending chiefly on the current produced by arrangements of different fluids have been devised. We append a general summary of the results.

(i). The sulphates formed with the following metals—K, Na, Mg, Mn, Fe, Ni, Co, Zn, Cu, Ag, &c.—among themselves obey Volta's tension law. We must except from this rule ammonium sulphate and sulphates of the formula $R_2(SO_4)_3$.

(ii). The haloid salts of potassium amongst themselves obey the tension law.

(iii). The acids do not generally follow the tension law, nor do solutions of salts in which the same base occurs combined with different acids.

(iv). Solutions of different concentration do not as a rule obey the tension law.

130. IV. *Electrical excitement between one or two metals and one or two fluids.—Theory of the voltaic cell.*

We are now in a position to consider the electrical state of a system consisting of two metals, say zinc aud platinum, partly immersed in a liquid, say hydrogen chloride.

On dipping the zinc plate into the fluid, a difference of potential Zn | HCl is established between them, and in dipping the platinum plate in, a difference Pt | HCl is established. The fluid being a conductor, a distribution of electricity over its surfaces takes place instantaneously, and

establishes equality of potential throughout the fluid mass. The zinc and copper plates are therefore at different potentials, the amount of difference being

$$Zn \mid HCl + HCl \mid Pt,$$

or since $Zn \mid HCl > Pt \mid HCl$ numerically, the platinum will be at a higher potential than the zinc. This difference could be tested by a quadrant electrometer, provided the alternate pairs of quadrants were of zinc and platinum respectively.

Suppose now a zinc wire laid across from the zinc to the platinum plate. At the point of contact with the platinum a new difference of potential is introduced represented by $Pt \mid Zn$.

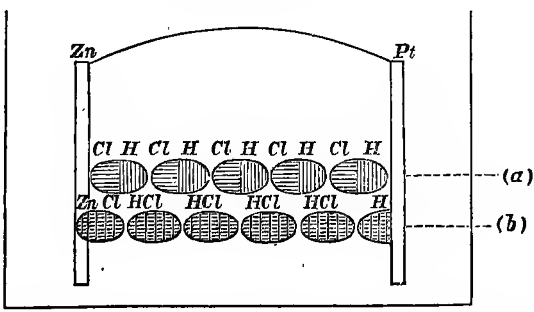
The whole difference of potential between the zinc plate and the other end of the zinc wire then becomes

$$Zn \mid HCl + HCl \mid Pt + Pt \mid Zn.$$

If the three substances followed Volta's law, this would necessarily vanish. Since however hydrogen chloride belongs to the second class of conductors (Art. 128), it will not vanish. Since the ends of the wire are now at different potentials, a flow of electricity takes place *through the wire* from the platinum towards the zinc plate, tending to equalize their potentials.

131. In consequence of this, the fluid in contact with the zinc acquires a higher potential than that in contact with

Fig. 41.



the platinum. The molecules of the fluid become polarized, having their positive ends turned towards the platinum,

and their negative ends towards the zinc. Hydrogen chloride being a compound body *we assume* that the elements hydrogen (H) and chlorine (Cl) exercise their own electrical affinities, the hydrogen being the electro-positive, and chlorine the electro-negative component. The arrangement of the compound molecules might be shown thus (fig. 41 *a*).

The chemical affinity of the zinc and chlorine now comes into play, causing the Zn to combine with the chlorine atoms next to it, so as to form zinc chloride ($Zn\,Cl_2$, two atoms of chlorine combining with each atom of zinc). The hydrogen of this molecule combines with the chlorine of the next, and so on along the whole row of molecules, leaving the hydrogen free at the platinum plate, the molecules at the same time each becoming neutral. This arrangement is shown in *b*, fig. 41. In this way the discharge of electricity has travelled round the whole circuit. The platinum plate is again brought to a higher potential than the zinc, and the same process is repeated, the successive discharges following each other with so great rapidity that their existence can only be inferred from theoretical considerations.

132. We find, however, apart from all theory, after the passage of the current for any length of time, that zinc is consumed, zinc chloride is formed in the cell, and hydrogen bubbles up at the platinum plate. So far our provisional theory accounts for the facts observed.

We find, moreover, that during the passage of the current heat is developed in all parts of the circuit, and that the conductor is capable of performing work external to itself (as the movement of a magnetic pole, for instance). We are in consequence compelled to look for a source of energy in the circuit. This source we find in the combination of zinc and chlorine. Whenever zinc chloride is formed, heat is evolved in the process, and it is found by actual experiment that the whole heat evolved (supposing no other work done) during the passage of the current is the same as that which would be given out by dissolving in Hydrogen chloride the amount of zinc that has combined with chlorine in the cell.

We see from this that chemical action is essential to the production of a current, and we infer that if we have the

metals X, Z, and a fluid Y, which cannot act on either of them chemically, there could be no current, or in other words,

$$X \mid Y + Y \mid Z + Z \mid X = 0;$$

or the substances obey Volta's tension series. For a galvanic cell then in its simplest form we must have at least one substance which belongs to the second class of conductors.

133. This peculiarity of decomposable fluids has been explained by saying that a metal in contact with a fluid exercises not only a *mass attraction*, but also an *atomic attraction*. The difference of potential between zinc and hydrogen chloride may be resolved into Zn | HCl the mass attraction, and [Zn | HCl] or [Zn | H + Zn | Cl] due to the attraction of the zinc for the separate atoms, the latter being denoted by being included in brackets. We might then write the whole difference of potential

$$\text{Zn} \mid \text{HCl} + [\text{Zn} \mid \text{HCl}] + \text{HCl} \mid \text{Pt} + [\text{HCl} \mid \text{Pt}] + \text{Pt}\,\text{Zn}.$$

We may now assume that, as far as the mass attractions are concerned, the substances obey Volta's law, so that

$$\text{Zn} \mid \text{HCl} + \text{HCl} \mid \text{Pt} + \text{Pt} \mid \text{Zn} = 0,$$

and the unbalanced difference of potential which originates the current is

$$[\text{Zn} \mid \text{HCl}] + [\text{HCl} \mid \text{Pt}],$$

due only to the atomic attraction of the metals on the elements of the fluid.

134. We must now define the terms of constant use in reference to a voltaic cell.

DEF. ELECTROMOTIVE FORCE *is used to denote the sum of all the differences of potential effective in a voltaic circuit.*

The term electromotive force is convenient, as we apply it to all cases in which a current is originated, even when we cannot strictly say that there is a difference of potential.

DEF. ELECTRODES. *The metal plates which dip into the fluid are called* electrodes, *that to which the external current flows being the* zincode, *and that from which it flows the*

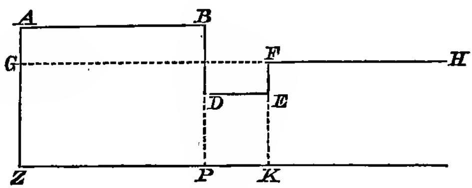
platinode. *The term is also extended to any two terminals from and to which electricity flows.*

DEF. POLES. *The term* pole *is used of the extremities of the conductor external to the fluid, that in connection with the platinode being the* positive pole, *that in connection with the zincode the* negative pole.

The direction of the current will therefore be in the fluid from the zincode to the platinode, and external to the fluid from the positive to the negative pole.

135. It is sometimes convenient to represent graphically the changes of potential in the course of a circuit. When the circuit is open this can easily be done provided we know in absolute measure the value of the successive differences that occur.

Fig. 42.



Thus, in the typical cell, let Z be the zinc plate, P the platinum plate, and K the junction of the zinc wire and platinum plate.

The differences of potential are

at Z, Zn | HCl, which will be positive, and may be represented by ZA;

at P, HCl | Pt, which will be negative and less than ZA, let it be BD;

at K, Pt | Zn, which will be positive and equal to EF suppose.

The broken line $ABDEFH$ gives us the law of change of potential throughout the circuit. The whole electromotive force of the cell is represented by

$$ZA - BD + EF = ZG \text{ suppose,}$$

and this would be the difference of potential of the two terminals or poles, as measured by a quadrant electrometer.

136. We say nothing here about the potential at any part of the circuit, which, if the cell be insulated, will be positive at one terminal and negative at the other. In practice, one part of the circuit is generally put to earth, and thus brought to zero potential. If the zinc plate be put to earth, then the ordinates in the figure represent the potentials, and ZG is the potential at the other end of the open circuit.

If the copper plate were put to earth the potentials would be shown by ordinates drawn to a horizontal line through DE, that at Z being equal to $-DP$ or $-EK$, and that at H to EF, the whole difference being in all cases equal to ZG.

137. The electromotive force of any cell can now be calculated *à priori*, if we know the differences of potential produced at the various contacts. The experimental difficulties render these determinations very unreliable, and we consequently content ourselves with knowing the whole electromotive force active in the cell, which is the only thing that practically concerns us. Having determined this for one cell in absolute measure, we can compare the electromotive forces of different cells with it, by methods to be explained further on.

138. V. *Electrical excitement due to gases condensed on the surface of metals. Polarization.*

It is well known that metals possess a remarkable power of condensing gases on their surface, and the electrical influence of these gases is seen in a variety of ways.

If two platinum plates be placed in hydrogen and oxygen gas respectively for some time, and be afterwards dipped in water (slightly acidulated to improve conduction), and then joined by a wire, a current is found to pass from the oxygen to the hydrogen plate. Since there is no contact of heterogeneous substances except platinum and water (which occurring twice, the differences should neutralize each other), the electromotive force must be due to a difference in behaviour towards water of a plate charged with oxygen and one charged with hydrogen. In this cell the electromotive force may be represented by

$$Pt_H \mid H_2O + H_2O \mid Pt_O,$$

where Pt_H and Pt_O denote respectively that the plate is

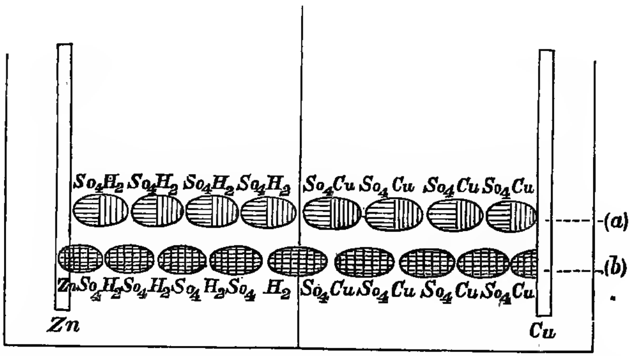
charged with hydrogen and oxygen. The passage of the current is accompanied by the disappearance of the free gases, which recombine to form water, and the cell is therefore active only for a short time. The energy of the cell was abstracted from the kinetic energy of the gases, and is equal to the energy of chemical separation of oxygen and hydrogen.

139. The same effect arises in all cells in which gas is liberated at the positive plate, unless the gas be soluble in the liquid round the plate. The liberated gas causes a backwards electromotive force which diminishes the effective electromotive force in the cell, and weakens the cell as soon as it is in action. This effect is commonly known as polarization.

140. To avoid this, a variety of cells have been constructed, in which the substance liberated at the positive plate is not gaseous, or if so, a gas which is soluble in the liquid which surrounds it.

In Daniell's cell there are two compartments divided by a membrane or a porous diaphragm, through which trans-

Fig. 43.



mission of fluid and chemical action takes place. In one compartment is placed a zinc rod, immersed in dilute sulphuric acid (H_2SO_4), and in the other a rod of copper immersed in copper sulphate ($CuSO_4$). In this cell the radical SO_4 (sulphion) takes the place of chlorine in the former cell, zinc sulphate being formed at the zinc plate, hydro-

gen sulphate at the diaphragm, while pure copper is deposited on the copper plate. The molecular arrangements during polarization and after discharge are shown in the rows of molecules *a*, *b* respectively.

The result of the action of the cell is that zinc is worn away, zinc sulphate being formed in the acid cell, while the copper sulphate is partly replaced by hydrogen sulphate in the salt cell, and copper is deposited on the copper plate or rod.

The electromotive force is represented by

$$Zn \mid H_2SO_4 + H_2SO_4 \mid CuSO_4 + CuSO_4 \mid Cu + Cu \mid Zn.$$

Groves' and Bunsen's cells illustrate the same principle. In them the nitrous oxide given off at the carbon or platinum plate being very soluble in nitric acid does not polarize the plate. These cells have a superior electromotive force to Daniell's cell, but are not so cleanly in working nor so durable.

141. The cells last alluded to are called constant, since the only limit to the working of the cell is apparently the exhaustion of some of the materials which compose it. There is another obstacle called *local action.* It is well known that commercial zinc contains impurities, and also that its density in different parts will be very different, while the production of pure and homogeneous zinc would be expensive, if not impossible. The consequence of this want of uniformity is to make the difference of potential between the zinc and fluid different at different parts of their commou surface, and galvanic circuits are set up through the zinc itself, which rapidly consume it, and interfere entirely with the action of the cell. To avoid this, the zinc plate used is rubbed over with mercury, which forms a pasty amalgam with the zinc, gives the latter a uniform surface for the action of the acid, and prevents the *local circuits.* The mercury itself is not attacked by the acid, but seems to improve the action of the cell by raising the difference of potential at the zinc plate.

By this means and the employment of various contrivances for ejecting the reduced zinc and supplying the other substances, batteries of the constant class can be kept in

working order (as for telegraph purposes) for months without further care than the occasional filling up with acidulated water.

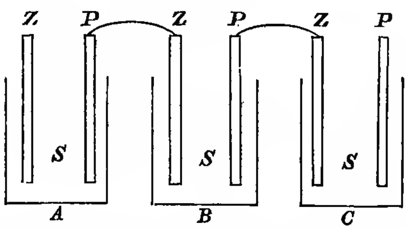
142. *Batteries.* The power of a galvanic cell may be increased to an unlimited extent by increasing the number of cells and arranging them in various combinations: the combination most suitable being determined by the circumstances of each particular case. It will be right here to consider the electromotive force in two arrangements, by compounding which all others are produced. These arrangements are—

1. *Compound circuit,* in which all the cells are arranged so that the platinode of one cell is joined to the zincode of the next, the circuit being completed by joining the zincode of the first cell to the platinode of the last.

2. *Simple circuit,* in which all the zincodes are joined to one terminal, and all the platinodes to another, the circuit being completed by joining these terminals.

143. I. *Compound circuit.* The arrangement with three cells, A, B, C, can be illustrated thus.

Fig. 44.



If S be the fluid and the zincode of A be to earth the potential at the zincode of B is

$$Z \mid S + S \mid P + P \mid Z \text{ or } E.$$

The rise of potential between the zincodes of B and C will be again E, making a total rise of $2E$.

Similarly, if there be n cells in compound circuit, the rise of potential in all the cells is nE. On this principle batteries have been constructed from which, without joining the terminals, sparks have been produced, Leyden jars charged, attraction and repulsion illustrated, and in fact all the phænomena of statical electricity exhibited. For these purposes several thousand cells must be joined in circuit, and each cell carefully insulated. For an account of these experiments consult Mr Gassiot's Memoir in *Phil. Trans. for* 1844.

144. Another illustration of the compound circuit is seen in Volta's crown of cups and in his pile. The last illustration (Fig. 44) is precisely his crown of cups.

In Volta's pile, a series of zinc and copper plates are arranged in the following order—

Zn S Cu Zn S Cu Zn S Cu..............Cu,

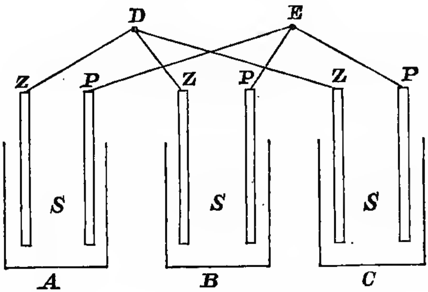
S denoting the fluid part of the circuit, which consists of pieces of flannel soaked in the fluid, generally acidulated water. The contiguous Cu Zn are soldered together to prevent the fluid soaking in between them. The theory of the pile is precisely the same as that of the compound circuit, the difference of potential of the terminals being simply proportional to the number of metal pairs.

145. To the same class belong the so-called *dry piles*, the best known of which is Zamboni's, used in the Bohnenberger Electroscope. In these piles there is an appearance of an electromotive force without a decomposible body. The fact seems to be that some of the elements of the pile (sheets of paper, for instance) are very hygroscopic, and perform the function of the fluid. These piles are found to suspend their action when thoroughly dried, and to regain it when left exposed to the damp of the air. In others glass or shellac seems to take the place of the fluid. With them the action of the pile improves on warming, and the reason seems to be that these substances when warm are decomposed by the circuit. That glass belongs to the second class of conductors is shown by the fact that when a current is passed through

two platinum plates immersed in molten glass, after the current has passed for some time, the plates exhibit the phænomena of polarization, and this can only happen, as far as is known, when the substance interposed is decomposed by the current.

146. II. *Simple circuit.* This arrangement may be illustrated as below,

Fig. 45.



all the zincodes being connected with a terminal D, and all the platinodes with a terminal E.

In this case all the zincodes are at the same potential, and all the platinodes at the same potential. The consequence is that the difference of potential between D and E will be only that due to a single cell, or will be simply

$$Z \mid S + S \mid P + P \mid Z.$$

This arrangement is in fact electrically identical with a single cell containing plates of three times the area, which of course in no way affects the electromotive force.

CHAPTER VI.

OHM'S LAW.

147. We have explained above in connection with Faraday's Theory of Induction, the state of a medium acted on by electrical forces (Art. 74). We then established

(i) That in each molecule there was a separation of electricity along the line of force through the molecule, the quantity separated being measured by $\pm \frac{F}{4\pi}$ per unit of area, where F is the resultant force at the point.

(ii) That this electrical separation produces or is produced by a strain in the medium along the lines of force, from which strained state the medium tends to return to a neutral state by a discharge from molecule to molecule through the medium.

(iii) That this discharge constitutes conduction resulting in a transfer of the positive electricity separated to the place of lower, and of the negative electricity separated to the place of higher potential, the lines of flow being the lines of force, and the quantity of electricity neutralized along a tube of force being $\pm \frac{1}{4\pi} F\sigma$.

148. The chief difference between the case there considered and our present problem is that here we have two parts of the conductor kept at constant potentials, so that as soon as one discharge has taken place, the strained state returns again owing to new separation of electricities, and

we get so rapid a series of discharges, that it cannot be distinguished from a continuous current.

149. We may still assume that the strain at any point in the conductor is measured by $\frac{1}{4\pi}F$, when F is the resultant force at the point, and since good or bad conduction consists only in easy or difficult transmission of electricity, the rate of flow at any point in a given body will be proportional to the force at that point. But experiment shows us that different bodies transmit the current in very different degrees, and consequently in different bodies the rate at which the series of charges and discharges succeed each other will be different. We shall assume therefore that the rate of flow at any point is measured by the product of a certain quantity c, which depends on the body and F, which measures the force of electrical separation at the point. The rate of flow at any point in the medium will therefore be measured by $c.F$.

150. Now to measure the rate of flow of a stream of water we should take a unit of area perpendicular to the stream-lines, and compute the quantity transmitted through it in a certain time. The same method applies in electricity, and we will suppose c so chosen, that cF measures the quantity transmitted per second across a unit of area of an equipotential surface over which the average force is F. If then a small tube of force be taken whose area is σ, the quantity transmitted per second across any section of the tube will be $cF\sigma$, and since $F\sigma$ is constant throughout the tube, the quantity transmitted per second, at whatever point in the tube it be measured, will be the same.

The quantity c depends entirely on the substance of the conductor, and is called its 'specific conductivity.'

DEF. SPECIFIC CONDUCTIVITY *of a substance may be measured by the quantity of electricity transmitted per second across a unit of area of an equipotential surface, at which the electric force has unit value.*

151. Again, if the conductor be bounded by a tube of force, the quantity transmitted along it per second will be

measured by $\Sigma cF\sigma$, and this quantity is called the 'strength of the current' in the tube.

DEF. STRENGTH OF CURRENT *in any tube of force is measured by the quantity of electricity transmitted per second along the tube of force.*

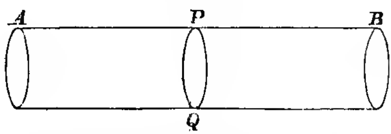
The principle shown above, that the strength of the current in all parts of a tube of force is the same, is often expressed by the phrase 'homogeneity of the circuit.' Since however this strength depends on c the current will not be homogeneous unless the substance of the conductor is the same throughout.

152. Since there is a transfer of the opposite electricities in opposite directions along the tube, it is impossible to speak strictly of the direction of the current, but as most of the phenomena depend on the directions assumed by the opposite currents, it is convenient to define the direction of flow of positive electricity as the direction of the current.

153. Prop. I. To find the strength of current in a conductor on which two surfaces bounded by closed curves are kept at constant potentials.

First, let the conductor be a cylinder whose two ends are at potentials V_1 and V_2.

Fig. 46.



Let the end A (Fig. 46) be at V_1 the higher potential, and the end B at V_2: also let l be the length, s the section, and c the specific conductivity of the cylinder.

Let I denote the quantity transmitted per second across any section PQ, or the strength of the current. When the difference of potential is first established, some of the lines of force will cut the surface of the cylinder, and a flow of electricity will take place along them, producing a superficial distribution, which combined with the constant electri-

fication of the ends, will make the lines of force parallel to the length of the cylinder. The tubes of force will then be cylindrical, the rate of change of potential along them constant, and the flow of electricity will be steady. The rate of change of potential or force along the cylinder will be equal to

$$\frac{V_1 - V_2}{l}.$$

Hence the quantity transmitted per second across any element of the section of the tube whose area is σ will be

$$cF\sigma = c \, . \, \frac{V_1 - V_2}{l} \, . \, \sigma.$$

Hence the whole quantity transmitted across a complete section will be

$$I = c\frac{V_1 - V_2}{l} s = \frac{cs}{l}(V_1 - V_2).$$

The quantity $\frac{cs}{l}$, which depends only on the substance and dimensions of the cylinder, is called its conductivity, and may clearly be defined as the quantity transmitted per second when the ends are at unit difference of potential.

154. *Secondly.* If the conductor be of any form.

The same reasoning may be extended to this case, as the first instantaneous effect of the flow of electricity is to produce a distribution on the surface such that every tube of force shall proceed from one to the other of the given surfaces, after which we have a steady flow of electricity along the tubes of force.

The amount transmitted through any tube is measured, as we have shown, by $cF\sigma$.

Now F is the rate of change of potential along a line of force, and, as shown (Art. 89), will at any point in the line be represented by $\kappa(V_1 - V_2)$, where V_1 and V_2 are the potentials at its ends. Hence if F_1 be the force at any point on the assumption that the two fixed surfaces are at unit difference of potential, $F_1 = \kappa$, and we have

$$F = F_1(V_1 - V_2),$$

and the amount of electricity transmitted through the tube per second is

$$cF_1\sigma\,(V_1 - V_2).$$

If now $F_1\sigma$ be computed for every tube of force which cuts through any given equipotential surface in the body, the whole amount of electricity transmitted per second will clearly be found by summing all the values so obtained. We shall have in fact the formula

$$I = c\,.\,\Sigma F_1\sigma\,.\,(V_1 - V_2).$$

155. It is clear that the coefficient $c\Sigma F_1\sigma$ depends only on the nature and geometry of the conductor. We call it the conductivity of the given conductor, as distinguished from the specific conductivity of its material defined above. We may give the physical definition as follows.

DEF. CONDUCTIVITY OF A CONDUCTOR *is the quantity of electricity which flows per second between two given surfaces on it which are kept at unit difference of potential.*

The numerical value of the conductivity is $c\Sigma F_1\sigma$, where F_1 is the force over the area σ on an equipotential surface, and c the specific conductivity. When the conductor is cylindrical, it is represented by $\dfrac{cs}{l}$, where s is the area of section, and l the length.

156. In practice we always use the *resistance* instead of the conductivity.

DEF. RESISTANCE *of a given conductor is numerically equal to the reciprocal of its conductivity.*

For any conductor its value is given by

$$\frac{1}{c\Sigma F_1\sigma}:$$

and for a cylindrical conductor by

$$\frac{1}{c} \times \frac{\text{length}}{\text{area of section}},$$

where c is the specific conductivity.

The quantity $\dfrac{1}{c}$ is often termed the *specific resistance* of the conductor.

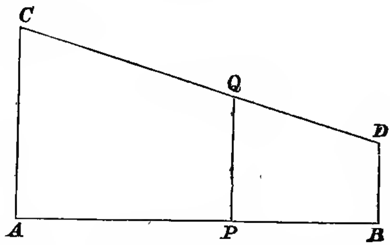
157. We can state now the proposition, that if two parts of a conductor be kept at potentials V_1 and V_2, and if R be the resistance of the conductor, and I the strength of the current,

$$IR = V_1 - V_2.$$

158. Prop. II. To represent graphically the law of change of potential in any cylindrical conductor whose ends are kept at certain given potentials.

Let the abscissa AB represent the resistance of the conductor, and at A, B set up ordinates AC and BD representing the potentials at those two points, and join CD.

Fig. 47.



At a point P in AB, draw an ordinate PQ. Then since $IR = V_1 - V_2$ for any portion of the circuit, it is clear that if V be the potential at P,

$$I = \frac{AC - V}{AP} = \frac{AC - BD}{AB};$$

hence $V = PQ$, the ordinate drawn to CD, or CD shows the law of fall of potential from A to B referred to resistance.

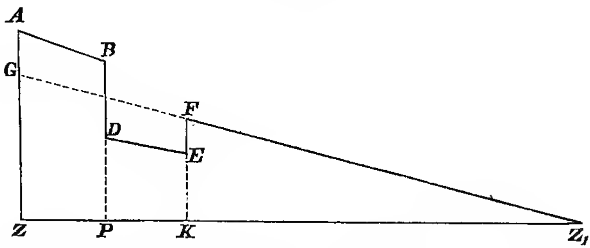
But when the conductor is cylindrical, the resistance is proportional to the length, and to each point on AB corresponds a point on the conductor dividing its length in the same ratio.

The same figure shows that we may graphically represent the current strength, $\frac{AC - BD}{AB}$, as the cotangent of the angle ACD, or as the tangent of the elevation of the line of potential.

159. Prop. III. In any voltaic circuit if E be the whole electromotive force and R, r the resistances of the conducting wire and fluid in the cell respectively, then the strength of current is given by $I = \frac{E}{R+r}$. Ohm's Law.

Let Z, P, K be the zincode, platinode, and junction of the two metals respectively, the abscissæ representing, as in the last Article, the resistances of the parts of the circuit solid and liquid. Owing to the changes in potential at the different contacts, the line of potential will be a broken line. The law of change we do not at present know, except at

Fig. 48.



the junctions, the potential at Z_1 being the same as at Z, the difference of potential at Z being denoted by $Z \mid S$, where S denotes the fluid, that at B by $S \mid P$, and that at K by $P \mid Z$.

Since the conductor throughout is not homogeneous, the circuit will not at first be homogeneous, but there will be a storing up of electricity at the different junctions. This storing up will bring the junctions to such potentials that the current strength between them is uniform. The current then becomes steady.

Let I represent its strength, and let V_1 be the potential in the fluid next the platinode, and V_2 the potential in the platinode wire at its junction with the zincode.

$$I = \frac{Z \mid S - V_1}{ZP} = \frac{V_1 - P \mid S - V_2}{PK} = \frac{V_2 + P \mid Z}{KZ_1}$$

$$= \frac{Z \mid S - V_1 + V_1 - P \mid S - V_2 + V_2 + P \mid Z}{ZP + PK + KZ_1}$$

$$= \frac{Z \mid S + S \mid P + P \mid Z}{ZP + PZ_1}.$$

But the numerator represents the electromotive force E, and the denominator the sum of the internal and external resistances. Hence we have

$$I=\frac{E}{R+r},$$

which is known as Ohm's law.

The formula also shows that the line of potential is in a constant direction, and its direction may be found by setting off as abscissa the whole resistance in circuit as ZZ_1, and as ordinate the whole electromotive force as ZG. The line GZ_1 gives us the law of fall of potential in the circuit, omitting of course the discontinuities at the various junctions.

160. Although the reasoning by which we have arrived at Ohm's law depends on molecular actions, which are assumed, but cannot be put to experimental test, the law itself has been subjected to the most rigorous experiment, and may be classed in point of certainty with the best ascertained physical laws.

161. In the formula $I=\frac{E}{R+r}$, there are three quantities which require to be measured, and it will be convenient here to remind the student of the units in which we have assumed them measured.

(i) *Electromotive force* is difference of potential, and its unit is the unit difference of potential, as defined in Chap. III.

(ii) *Current strength* is the quantity of electricity transmitted per second along a conductor, and its unit will be the strength of a current sending a unit of quantity per second.

(iii) *Resistance* is a new idea, and must be measured in accordance with the above formula by the resistance of a conductor, which allows a unit of electricity to flow per second through it, the two ends being kept at a unit difference of potential.

162. These are the units which we have used in theory, but they would be very inconvenient in practice. The practical units depend on electromagnetic phenomena, and we

must defer their precise definition till we come to that part of our subject. We will merely state now that

(i) *Electromotive force* is measured by the *volt*, which is about equal to that of a Daniell's cell.

(ii) *Resistance* is measured by the *ohm*, which is a certain coil of wire offering a definite resistance.

(iii) *Current strength* is measured by a *farad per second*, the farad being the quantity transmitted per second through a circuit in which the electromotive force is one volt, and the total resistance one ohm.

We shall assume in future, unless *absolute* units are referred to, that quantities are measured in these terms. For measuring them we require in practice a galvanometer, a set of resistance coils, and a cell whose electromotive force is known.

163. Prop. IV. To find the current strength when n cells each of resistance r and electromotive force E are arranged in simple circuit (see **Art. 146**).

We have already shown that in this case the electromotive force is unaltered, the arrangement being equivalent to a single cell in which the plates are n times as large.

Again, internal resistance is measured by

$$\frac{1}{c} \cdot \frac{\text{length}}{\text{section of area}}.$$

Here length is the distance of the plates, and since there are n plates, the surface in contact with fluid is n times as great. Hence the resistance is only $\frac{r}{n}$. This formula therefore becomes

$$I = \frac{E}{\frac{r}{n} + R} = \frac{nE}{r + nR}.$$

Cor. 1. When the internal resistance is small compared to the external, this formula is equivalent to $I = \frac{E}{R}$, and the current strength is not increased by increasing the number of cells. If, however, R be small compared to r, or the

external resistance be very small, the formula is equivalent to $I = \frac{nE}{r}$, or the current strength is increased in proportion to the number of cells.

164. Prop. V. To find the current strength when n cells are arranged in compound circuit. (Art. 143.)

Here we have shown that the whole electromotive force is nE. Each cell, however, introduces a fresh resistance, and the whole resistance in the battery becomes nr.

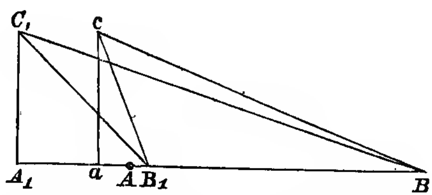
Hence Ohm's formula gives

$$I = \frac{nE}{nr + R}.$$

COR. If r is small compared to R, or when the internal resistance is small $I = \frac{nE}{R}$, or the current strength is increased n-fold. But if nr be large compared to R, the formula reduces to $\frac{nE}{nr} = \frac{E}{r}$, or the current is not increased by increasing the number of cells.

165. Remembering the construction for the line of potential, we can illustrate graphically the two last propositions.

Fig. 49.



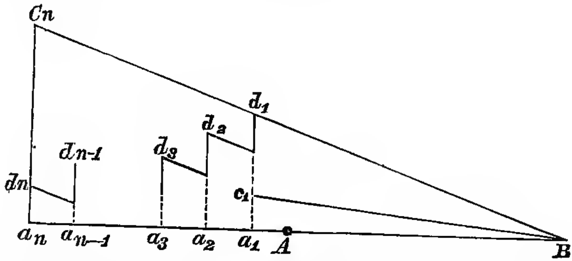
Let AA_1 be the internal resistance, and A_1C_1 the electromotive force of a single cell. If there be n cells simple circuited, the internal resistance is $Aa = \frac{1}{n} AA_1$, and $ac = A_1C_1$.

Then if the external resistance be small compared to AA_1 (as AB_1) the intensity is increased in ratio tan AB_1c to tan AB_1C_1.

But if the external resistance be several multiples of AA_1 as AB, the increase is only in the ratio tan ABc to tan ABC_1 nearly an equality, or the simple circuit is almost useless.

166. In the case of a compound circuit, the resistances of the successive cells are represented by Aa_1, a_1a_2, $a_2a_3 \ldots a_{n-1}a_n$, and the electromotive force is $a_nC_n = n \,.\, a_1c_1$, or n times the electromotive force of a single cell. If AB be the external resistance, the line of potential will *outside the battery* be given by Bc_n. For the law of change in the battery we clearly have a broken line discontinuous at each junction, but if we assume the whole rise in each cell to take place at the zinc plate, assuming the last zinc in connection with B, the line will be represented by the broken line $d_1d_2 \ldots\ldots d_n$.

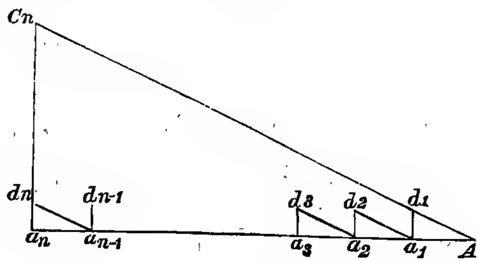
Fig. 50.



If a_1c_1 represent the electromotive force of a single cell, the increase in quantity will be in the ratio tan ABc_n to tan ABc_1. This is large when AB is considerable.

167. If AB be very small so that A and B coincide, the line of potential will be as in the following figure, in which,

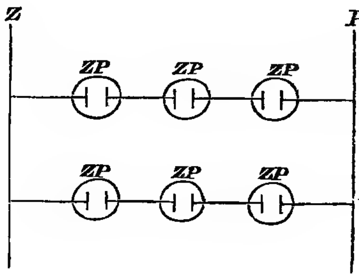
Fig. 51.



from the construction of the figure, it is clear that the fall in each cell of the battery is equal to the rise at each zinc plate. There is in this case no gain from using a compound circuit.

168. The corollaries to the two last propositions show that when the external resistance is very large there is no advantage obtained by arranging the cells in simple circuit, and when very small there is no advantage in arranging them in compound circuit. When the resistance is moderate we obtain a greater current than by either arrangement, by a combination of the two, which may be called *mixed circuit*, and is illustrated in the six cells following, in which the

Fig. 52.



vertical rows are simple circuited, and the horizontal rows compound circuited; the arrangement being the same as that of three cells arranged in compound circuit, each cell having plates twice as large as those of the given cell.

169. Prop. VI. To find the current-strength due to pq cells arranged in q horizontal rows of p cells, the cells in each row being in compound circuit and the successive rows in simple circuit.

Here the electromotive force is clearly pE, and the resistance in the battery $p \times \frac{r}{q}$, since the arrangement is the same as that of p cells whose plates are q times as large as the plates of each cell.

Hence Ohm's formula gives

$$I=\frac{pE}{\frac{p}{q}r+R}$$

170. Prop. VII. To find the best arrangement of n cells when the external resistance is given.

Let them be arranged in q rows of p cells each.

Then we have $$n=pq,$$

and $$I=\frac{pE}{\frac{p}{q}r+R}.$$

We want to find values of p and q which make I a maximum.

We have $$I=\frac{pE}{\frac{p}{q}r+R}=\frac{E}{\frac{r}{q}+\frac{R}{p}}.$$

Hence we must make $\frac{r}{q}+\frac{R}{p}$ as *small* as possible.

$$\text{Now } \frac{r}{q}+\frac{R}{p}=\frac{pr}{n}+\frac{R}{p}$$
$$=2\sqrt{\frac{Rr}{n}}+\left(\sqrt{\frac{pr}{n}}-\sqrt{\frac{R}{p}}\right)^2.$$

The right-hand side is a *minimum* when the square it contains vanishes.

Hence, if *R is not too great*, we make

$$\sqrt{\frac{pr}{u}}=\sqrt{\frac{R}{p}},\quad \text{or } R=\frac{p^2r}{u}=\frac{pr}{q},$$

or the external resistance must be made equal to the internal resistance.

The greatest value of $\frac{pr}{q}$ is clearly when $p=n$ and $q=1$; then $R=nr$. If R has this or any greater value, the compound circuit is the best. If R is less than this we must choose p and q so as to satisfy as nearly as possible the condition of making the external and internal resistances equal.

171. The proposition of the preceding article may also be treated graphically.

If there be n cells of electromotive force E and resistance R, arranged in q rows of p cells, we have for ϵ the electromotive force and ρ the internal resistance of the battery, the equations

$$\epsilon = pE,$$

$$\rho = \frac{pR}{q},$$

$$n = pq.$$

Hence $$\rho = \frac{p^2 R}{n} = \frac{R}{n}\left(\frac{\epsilon}{E}\right)^2;$$

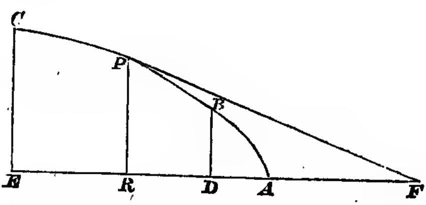
$$\therefore \epsilon^2 = \frac{nE^2}{R} \cdot \rho;$$

an equation which shows that if ρ be an abscissa and ϵ the corresponding ordinate, the locus of its extremity is a parabola whose latus rectum is $\frac{nE^2}{R}$. Hence in all arrangements of the battery the relation between its internal resistance and electromotive force is represented graphically by the abscissa and ordinate of a parabola.

The only part of the curve practically available will be that between the abscissa $\frac{R}{n}$, when the cells are simple circuited, and nR when they are compound circuited.

Tracing the curve we shall have the portion between B and C, for instance, available.

Fig. 53.



If the external resistance be set off to the right of A along the axis, equal suppose to AF, the intensity of the current when simple circuited is $\tan AFB$, and when compound circuited $\tan AFC$ (Art. 158).

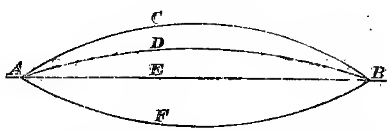
The greatest possible strength of current will be that corresponding to a tangent drawn from F to the parabola, suppose FP; then AR is the internal resistance, PR the electromotive force, and the intensity is given by $\tan RFP$.

By a well-known property of the parabola we have $AR = AF$, or the external and internal resistances are equal.

172. Prop. VIII. To investigate the strength of the current and the whole resistance in any divided circuit.

Suppose the potentials at two points A, B to be V and V'. Let the resistances in the various branches ACB, ADB, AEB, &c. be R_1, R_2, R_3 ... and the current strengths I_1, I_2, I_3....

Fig. 54.



Then in the respective branches by Prop. I.

$$V - V' = I_1R_1 : \ V - V' = I_2R_2 : \ V - V' = I_3R_3, \ \&c.$$

Hence
$$V - V' = \frac{I_1}{\frac{1}{R_1}} = \frac{I_2}{\frac{1}{R_2}} = \frac{I_3}{\frac{1}{R_3}} = \ldots\ldots$$
$$= \frac{I_1 + I_2 + I_3 + \ldots\ldots}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots}.$$

But the whole current passing is clearly the sum of that passing in each branch.

Hence $$I = I_1 + I_2 + I_3 + \ldots\ldots$$

Let also $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots$$

Hence $$V - V' = \frac{I}{\frac{1}{R}} = IR.$$

But by Art. 157, when

$$I = \frac{V - V'}{R},$$

R is by definition the resistance of the conductor.

Hence for the resistance of any divided circuit we have, if R be the whole resistance and R_1, R_2... the branch resistances,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots\dots$$

and the current in each branch is inversely proportional to its resistance.

173. Prop. IX. To investigate the current in each branch of any net-work of linear conductors.

Any net-work may be resolved into a system of linear conductors, and a system of junctions at which three or more linear conductors meet.

We must begin by fixing arbitrary values to the potential of each junction, except two, at which we must suppose the potentials given by connection with a battery or otherwise.

For each linear conductor at whose extremities the potentials are V_r and V_p, whose resistance is R_s, and in which the current is I_s, we have

$$V_r - V_p = I_s R_s \dots\dots\dots\dots\dots\dots\dots (A),$$

and similarly for each linear portion.

For each junction we know that the same amount of electricity which flows to it must flow from it. Hence if I_1, I_2, I_3... be the intensities of the currents flowing *all to or all from* a given junction, we have the *algebraical equation*

$$I_1 + I_2 + I_3 + \dots\dots = 0,$$

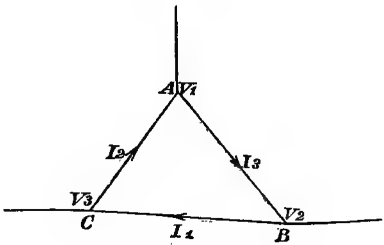
$$\text{or} \quad \Sigma I = 0 \dots\dots\dots\dots\dots\dots\dots (B).$$

The systems of equations (A), (B) always give us enough simple simultaneous equations to find the current in each branch and the potential at each junction.

174. The equations (A) can usually be simplified by regarding the net-work as a set of closed circuits.

In any closed circuit ABC, where there is no impressed electromotive force, we have

Fig. 55.



$$V_3 - V_1 = I_2 R_2,$$
$$V_1 - V_2 = I_3 R_3,$$
$$V_2 - V_3 = I_1 R_1.$$

Hence $$I_1 R_1 + I_2 R_2 + I_3 R_3 = 0,$$

$$\text{or} \quad \Sigma IR = 0.$$

If there be in any branch an impressed electromotive force E, we have similarly

$$\Sigma IR = E \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

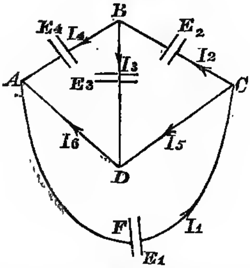
The sets of equations (B), (C) will be generally sufficient to determine the current in each branch.

175. Prop. X. To investigate the current-strength in a system consisting of six conductors joining four points (a quadrilateral and its diagonals), four of the branches having in them electromotive forces.

Let $ABCD$ represent such a system. Let the current-strengths in the branches be $I_1, I_2 \ldots\ldots I_6$, the electromotive

forces $E_1 \ldots\ldots E_4$, and the total resistances $R_1 \ldots\ldots R_6$, as in figure.

Fig. 56.



The equations for current-strengths are—

$$\left.\begin{array}{lll} \text{from } ABCD; & E_2 + E_4 = I_2 R_2 + I_4 R_4 - I_5 R_5 - I_6 R_6 & \ldots\text{(i)} \\ \text{from } ABD; & E_3 - E_4 = I_3 R_3 - I_4 R_4 + I_6 R_6 & \ldots\ldots\ldots\ldots\text{(ii)} \\ \text{from } ADCF; & E_1 = I_1 R_1 + I_5 R_5 + I_6 R_6 & \ldots\ldots\ldots\ldots\text{(iii)} \end{array}\right\} (C);$$

$$\left.\begin{array}{lll} \text{at } A; & 0 = I_1 - I_4 - I_6 & \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)} \\ \text{at } B; & 0 = I_2 - I_3 - I_4 & \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)} \\ \text{at } C; & 0 = I_1 - I_2 - I_5 & \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)} \end{array}\right\} (B).$$

The six equations (B), (C) are independent and sufficient to determine the current-strength in each of the six branches.

An important case arises when the current in one branch is independent of the electromotive force and resistance in another branch, in which case these two branches are said to be *conjugate* to each other.

176. Prop. XI. To show that in the system of six conductors joining four points the diagonals are conjugate to each other if the products of the resistances in the opposite sides be equal.

We must proceed to solve the six equations of the last proposition to find I_3. We shall adopt the method of indeterminate multipliers, multiplying the equations (ii)—(vi) by $\lambda_1, \lambda_2 \ldots \lambda_5$ respectively. If we add the resulting equations together and equate separately to zero the coefficients of I_1, I_2, I_4, I_5, I_6, we shall have five equations to determine $\lambda_1 \ldots \lambda_5$, and the remaining equation for I_3—

$$I_3 (\lambda_1 R_3 - \lambda_4) = E_2 + E_4 + \lambda_1 (E_3 - E_4) + \lambda_2 E_1 \ldots\ldots\text{(vii)}.$$

I_3 will by this equation be independent of E_1 if $\lambda_2 = 0$.

Writing down the five equations for λ's, with the extra condition $\lambda_2 = 0$, we have

$$\left.\begin{aligned} \lambda_3 + \lambda_5 &= 0 \\ R_2 + \lambda_4 - \lambda_5 &= 0 \\ R_4 - \lambda_1 R_4 - \lambda_3 - \lambda_4 &= 0 \\ -R_5 - \lambda_5 &= 0 \\ -R_6 + \lambda_1 R_6 - \lambda_3 &= 0 \end{aligned}\right\} \text{..............(viii)}.$$

The condition that these equations can be satisfied simultaneously is found by eliminating λ_1, λ_3, λ_4, λ_5 from them. The result is easily seen to be

$$R_4 R_5 - R_2 R_6 = 0 \text{..........................(ix)}.$$

If (ix) is satisfied, (viii) can be satisfied, and (vii) will then be satisfied with the extra condition $\lambda_2 = 0$. In this case I_3 is independent of E_1, and since R_1 enters none of equations (vii), (viii), I_3 must be also independent of R_1.

Cor. 1. Conversely the current in AC will, if (ix) be satisfied, be independent of the electromotive force and resistance in BD.

Cor. 2. If $I_3 = 0$, which will happen if (ix) be satisfied and E_1 be the only electromotive force, it is clear that R_3 enters none of the equations (B), (C), and the currents in all the branches will be unaltered by making or breaking contact in BD.

177. Prop. XII. To find the time of discharge of a given electrified system.

Let the two surfaces A, B be at potentials V_1 and V_2, and let R be the resistance of the medium interposed between them, all measured in absolute units. Let also C be the electrostatic capacity of the system; then the quantities of electricity $\pm C(V_1 - V_2)$ tend to neutralize each other by conduction through the medium.

Let us assume that v is the difference of potential, and $\pm q$ the quantities of electricity after a time t, and that v' and $\pm q'$ represent the same things after $t + \tau$, where τ is a very short interval.

By Ohm's law $I = \dfrac{v}{R}$, where I is quantity of flow per second.

Hence the quantity which flows through in time τ

$$= I\tau = \frac{v\tau}{R}.$$

Hence $$q - q' = \frac{v\tau}{R}.$$

But $$q - q' = C(v - v');$$

$$\therefore\ CR(v - v') = v\tau.$$

Hence $$\tau = CR\frac{v - v'}{v}.$$

As before in Art. 113, when $\dfrac{v - v'}{v}$ is very small, as will be the case here, we may put

$$\frac{v - v'}{v} = -\log\left(1 - \frac{v - v'}{v}\right)$$

$$= -\log\frac{v'}{v};$$

$$\therefore\ \tau = +CR\log\frac{v}{v'}$$

$$= CR(\log v - \log v').$$

The same proposition will hold for any number of very short intervals; we shall have, if v be the difference of potential after a time t from charging,

$$t = CR\{\log(V_1 - V_2) - \log v\};$$

$$\therefore\ \frac{t}{CR} = \log\frac{V_1 - V_2}{v};$$

$$\therefore\ v = (V_1 - V_2)\,\epsilon^{-\frac{t}{CR}},$$

which gives the potential at any time t.

If we observe that $v = \dfrac{1}{n}(V_1 - V_2)$, or the difference of potential falls to one n^{th} of its first value,

$$\frac{1}{n} = \epsilon^{-\frac{t}{CR}},\ \text{or}\ t = CR\log n.$$

178. Prop. XIII. If K be the specific inductive capacity, and ρ the specific resistance of a substance, and if C be the electrostatic capacity of any condenser made of that substance, and R its resistance to the passage of electricity; to prove that $CR = \frac{1}{4\pi} \cdot \rho K$ in absolute measure.

It is assumed that lines of force proceed exclusively from one surface to the other of the body under consideration.

To find the electrostatic capacity, we notice that along any tube of force $F\sigma$ is constant, and at either bounding surface $F\sigma = 4\pi\rho\sigma = 4\pi q$. Hence the whole charge Q on either surface is given by

$$4\pi Q = \Sigma F\sigma,$$

and if F_1 be computed on the assumption that the opposite surfaces differ in potential by one unit,

$$Q = C \text{ when } F = F_1;$$

$$\therefore \; 4\pi C = \Sigma F_1\sigma,$$

if the dielectric be air. In the supposed case where the dielectric has specific inductive capacity K,

$$C = \frac{K}{4\pi} \Sigma F_1\sigma.$$

But it has already been shown (Art. 156)

$$R = \frac{1}{c\Sigma F_1\sigma} = \frac{\rho}{\Sigma F_1\sigma},$$

where $\frac{1}{\rho} = c$, the specific conductivity;

$$\therefore \; C = \frac{K}{4\pi} \cdot \frac{\rho}{R},$$

$$\text{or} \quad RC = \frac{1}{4\pi} \rho K.$$

COR. We have shown in the last article that if t be the time of falling to $\left(\frac{1}{n}\right)^{\text{th}}$ of charge,

$$RC = \frac{t}{\log_e n}.$$

Hence we have

$$RC = \frac{1}{4\pi}\rho K = \frac{t}{\log_e n}.$$

179. Prop. XIV. To calculate the amount of Heat developed in any portion of a galvanic circuit.

Let the potentials in absolute measure at the extremities of the circuit be V_1 and V_2, R the resistance of the interposed circuit, and I the strength of the current. By definition of current-strength I units of electricity pass from potential V_1 to potential V_2 per second, and when no external work is done, this amount of energy must be converted into heat in the circuit.

Hence the mechanical equivalent of the heat given out per second is $I(V_1 - V_2)$.

Again, if J represent Joule's mechanical equivalent of heat, or the number of ergs imparted to a gramme of water which is warmed from 0°C. to 1°C., and if H be the number of units of heat given out per second, then JH will also represent the mechanical equivalent of the heat given out per second, and we have

$$JH = I(V_1 - V_2).$$

But $$V_1 - V_2 = IR.$$

Hence $$JH = I^2 R = \frac{(V_1 - V_2)^2}{R}.$$

If the conductor be a wire whose specific heat is c, w its weight in grammes, and θ the elevation of temperature, then $H = cw\theta$;

$$\therefore\ Jcw\theta = I^2 R = \frac{(V_1 - V_2)^2}{R} = I(V_1 - V_2),$$

a formula giving the elevation in temperature owing to the passage of the current.

Cor. 1. It appears from the last formula that the elevation in temperature is independent of the length of the wire, provided the intensity of the current is constant, for θ will be proportional to $\frac{R}{w}$, which is a constant, since both numerator and denominator are proportional to the length.

COR. 2. If the section of the wire vary, the current remaining of the same strength, it is clear that R varies inversely as the area of section, and w varies directly as the area of section. Hence the quotient $\frac{R}{w}$ will vary inversely as the square of the section, or if the section be similar throughout, inversely as the fourth power of the diameter.

180. Prop. XV. To show that in any divided circuit in which there is a distribution of the current-strength in each branch inversely as its resistance, there will be less heat given out than if the same total currents were distributed in any other way. Principle of Least Heat.

First, let the circuit contain only two branches. Then if I_1, R_1 and I_2, R_2 be corresponding quantities for the two branches, and I the whole current,

$$I = I_1 + I_2,$$

and
$$JH = I_1^2 R_1 + I_2^2 R_2;$$

$$\begin{aligned}\therefore\ JH(R_1 + R_2) &= I_1^2 R_1^2 + I_2^2 R_2^2 + (I_1^2 + I_2^2) R_1 R_2 \\ &= (I_1 R_1 - I_2 R_2)^2 + (I_1 + I_2)^2 R_1 R_2 \\ &= (I_1 R_1 - I_2 R_2)^2 + I^2 R_1 R_2.\end{aligned}$$

The right-hand side will have the least possible value when the first term vanishes, or when

$$I_1 R_1 = I_2 R_2,$$

i.e. the current in each branch is inversely as the resistance.

Hence also H or the heat given out will have its least possible value.

Secondly, in any divided circuit we see that for any two of the branches this relation must hold, or we could redistribute the current in these two so as to evolve less heat without disturbing the current in the other branches. Hence we infer, however many branches there be, there will be least heat evolved when the current in each varies inversely as the resistance, or when the currents are distributed according to Ohm's law.

CHAPTER VII.

PROBLEMS IN VOLTAIC ELECTRICITY.

181. In the following problems we shall endeavour to illustrate the propositions of the preceding Chapter by laying before the student a number of results mostly of the highest importance to the practical electrician.

The chief instruments we shall assume used will be a galvanometer and a box of resistance coils, with perhaps occasionally a quadrant or other form of electrometer capable of giving absolute measure. The theory of the galvanometer we do not enter into here as it belongs to Magnetism. We shall assume however that the form used is that known as the tangent galvanometer (unless the contrary be stated), in which the strength of the current is proportional to the tangent of the deflection.

It is not generally necessary to determine a current in absolute measure, our problems nearly always depending on the comparison of two currents with the same galvanometer.

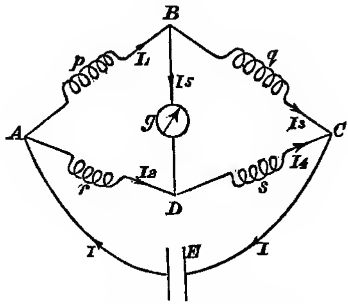
182. Prop. I. To investigate the electrical conditions of Wheatstone's Bridge.

Wheatstone's Bridge is only a particular case of the system of conductors investigated in Arts. 175, 176.

This instrument consists essentially of a double divided circuit, two points in the divided branches being joined by a conducting wire. These divided circuits are ABC and ADC, and BD is the joining wire. In the portions AB, BC, CD, DA, are introduced resistances, which we shall call p, q, s, r, and in BD is a galvanometer whose resistance we call g. The current from a galvanic cell enters at A and leaves at C. The

cell is E, and we shall denote its electromotive force by E and resistance by R.

Fig. 57.



The current-strengths in the various branches we denote by I_1, I_2, I_3, I_4, I_5, I_6, as shown in figure.

Then in circuit $EABCE$ we have

$$E = RI + pI_1 + qI_3 \quad \text{(i)},$$

in ABD,
$$0 = pI_1 + gI_5 - rI_2 \quad \text{(ii)},$$

in BCD,
$$0 = qI_3 - sI_4 - gI_5 \quad \text{(iii)},$$

at A,
$$I = I_1 + I_2 \quad \text{(iv)},$$

at C,
$$I = I_3 + I_4 \quad \text{(v)},$$

at B,
$$I_1 = I_5 + I_3 \quad \text{(vi)}.$$

These six equations will be found independent, and can be easily solved, giving the strength of the current in each of the six branches.

By (vi), $\qquad I_3 = I_1 - I_5$.

By (v) and (iv), $I_4 = I - I_3 = I - I_1 + I_5 = I_2 + I_5$.

Substitute in (iii),

$$0 = q(I_1 - I_5) - s(I_2 + I_5) - gI_5,$$

or
$$qI_1 - sI_2 - (q + s + g) I_5 = 0.$$

By (ii), $\qquad pI_1 - rI_2 + gI_5 = 0.$

Hence $$\frac{I_1}{r(q+s+g)+sg}=\frac{I_2}{p(q+s+g)+qg}=\frac{I_5}{qr-ps},$$

$$\therefore\ =\frac{I_1-I_5}{s(p+r+g)+rg}=\frac{I_3}{s(p+r+g)+rg}\text{ by (vi)},$$

also $$=\frac{I_2+I_5}{p(q+g)+q(r+g)}=\frac{I_4}{p(q+g)+q(r+g)}\text{ by (v)},$$

and

$$=\frac{I_1+I_2}{(r+p)(q+s)+g(p+q+r+s)}=\frac{I}{(r+p+g)(q+s+g)-g^2}.$$

These equations with (i) give the currents in the branches.

The one of practical use is I_5, the current in the galvanometer branch, and in the general use of the instrument this is made to vanish. The condition for this is clearly that its denominator shall vanish, or that

$$qr-ps=0,$$

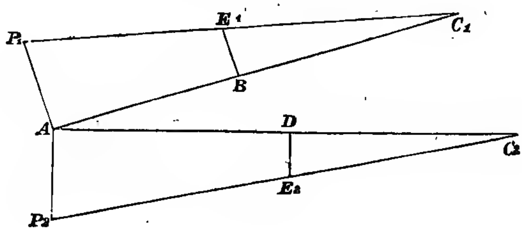
or $$p:q::r:s,$$

a relation on which the use of the bridge depends.

183. In practice one resistance is generally known, that in AB suppose, while that in BC is required to be determined. ADC generally consists of a straight wire at any point of which by a key the branch BD is completed. By moving the key we find the point D for which the current vanishes, and reading off the relation of the lengths $AD:DC$, we have $$AD:DC::p:q,$$ whence q becomes known.

184. We might at once see by the graphical method that if this relation hold, there is no current in the galvanometer.

Fig. 58.



Let the resistances ABC and ADC be represented by ABC_1, ADC_2.

Let the differences of potential between the extremities of the conductor be represented by AP_1 perpendicular to AC_1, and AP_2 equal to it perpendicular to AC_2. The line P_1C_1 represents the fall of potential along AC_1, and P_2C_2 along AC_2.

By similar triangles

$$BE_1 : AP_1 :: C_1B : C_1A,$$

and
$$DE_2 : AP_2 :: C_2D : C_2A.$$

Hence
$$BE_1 = DE_2,$$

if
$$C_1B : C_1A :: C_2D : C_2A,$$

or
$$C_1B : BA :: C_2D : DA,$$

or
$$C_1B \,.\, DA = BA \,.\, C_2D,$$

the relation already found. If B and D be now joined there will be no current in BD, since the extremities are at the same potential.

Cor. It follows that the currents in the other branches will remain unaltered whether the branch BD be open or closed.

185. Prop. II. To find the resistance of a galvanometer coil.

This resistance can be measured by Wheatstone's Bridge just as any other conductor. After the magnet has been mounted in its place, the following method, due to Sir W. Thomson, is found to lead to more accurate results.

Place the galvanometer in the branch BC (fig. 57), and in BD place a contact-breaker instead of a galvanometer. It appears by the corollary to the last Proposition that if the relation $ps = rq$ be satisfied, the galvanometer deflection will remain the same whether contact in BD be made or broken.

We have therefore only to adjust the other resistances until the galvanometer reading does not alter on making or breaking contact in BD, and then the resistance of the galvanometer is given by

$$q = \frac{sp}{r}.$$

186. Prop. III. To measure the internal resistance of a Battery.

1*st Method.* If we make circuit in the battery by pieces of stout wire connecting its poles with the galvanometer, the only external resistance will be that of the galvanometer, g. Then if δ_1 be the observed deflection, and x the unknown internal resistance,

$$c \tan \delta_1 = \frac{E}{x+g},$$

where c depends on the galvanometer.

Introduce now between one pole and the galvanometer a measured resistance r. If δ_2 be the new deflection

$$c \tan \delta_2 = \frac{E}{x+r+g},$$

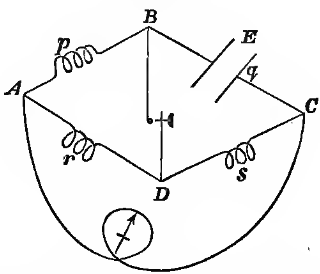
dividing
$$\frac{\tan \delta_1}{\tan \delta_2} = \frac{x+r+g}{x+g},$$

a simple equation for x.

This method is open to many objections, as the observations are taken with two different current-strengths. From these objections Mance's method seems free.

187. 2*nd Method* (Mance's). In this method is employed a modification of Wheatstone's Bridge, similar to that used

Fig. 59.



by Sir William Thomson for measuring the galvanometer resistance. The cell whose resistance is to be measured is placed in BC, the galvanometer in AC, and a contact-breaker in BD. The arrangement will then be as in the figure.

Now if the resistance in the branches satisfy the condition $ps = qr$, the branches BD and AC are conjugate, and consequently making or breaking contact in BD will produce no effect on the galvanometer in AC. If therefore one resistance be adjustable, we adjust it until the galvanometer is uninfluenced by making and breaking, and we have then for q, the unknown resistance,

$$q = \frac{ps}{r}.$$

188. Prop. IV. To compare the electromotive force of two cells.

1st *Method.* Take one cell and introduce resistance till the galvanometer stands at a certain deflection δ_1. Let r_1 be the resistance introduced; g, R those of the galvanometer and cell; then

$$c \tan \delta_1 = \frac{E_1}{r_1 + g + R}.$$

Add resistance r_1', so that the deflection comes down to δ_2,

$$\therefore\ c \tan \delta_2 = \frac{E_1}{r_1' + r_1 + g + R};$$

$$\therefore\ \frac{1}{c} (\cot \delta_2 - \cot \delta_1) = \frac{r_1'}{E_1}.$$

Next, by introducing resistance into the circuit of the other cell bring its deflection to δ_1. Add resistance (suppose r_2') till its deflection is δ_2 as before.

Then
$$\frac{1}{c} (\cot \delta_2 - \cot \delta_1) = \frac{r_2'}{E_2};$$

$$\therefore\ \frac{r_1'}{E_1} = \frac{r_2'}{E_2};$$

$$\therefore\ E_1 : E_2 :: r_1' : r_2',$$

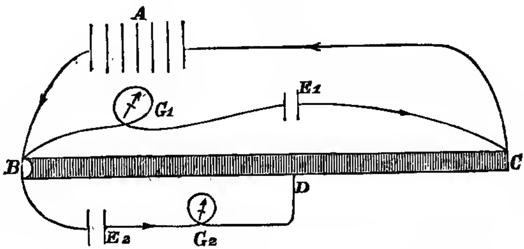
or the electromotive forces are proportional to the resistances which must be introduced to bring the galvanometer from one fixed reading δ_1 to another δ_2.

The objection to this simple method is that the electromotive forces are subject to variation from a variety of causes when the battery is in action, and the comparison should always be made when no current is passing.

189. *2nd Method. By Clark's Potentiometer.* For this method we require a battery of very constant electromotive force, and a length of fine wire coiled along an ebonite cylinder similar to Wheatstone's rheostat.

Let A be the constant battery, BC the cylinder, and E_1, E_2 the cells to be compared.

Fig. 60.



Connect the battery A and a variable resistance in the circuit ABC, and make a branch circuit BG_1E_1C containing a galvanometer G_1 and the cell E_1 (which is supposed to have greater electromotive force than E_2), so placed that its current in BC is opposite in direction to that of A. Vary the resistance in AC till the galvanometer G_1 is at zero, when the difference of potential between B and C will equal E_1. Introduce now the second cell E_2 having its negative pole at B, with a second galvanometer G_2. With the positive pole D make contact at successive places along BC till there is no current in G_2. We then have $E_2 : E_1 :: BD : BC$. Hence if a divided scale be attached to BC and graduated from 0 to 100, we can at once read off the electromotive force of E_2 in terms of E_1.

190. Prop. V. To find the position of a "fault."

In practical telegraphy faults arise from a large variety of causes and under a variety of circumstances, which influence considerably the method adopted for their detection and the determination of their position.

The following include a few of the methods most commonly applied.

1*st Method.* For a land-line in which the wire is completely broken, the broken end not making earth.

In this case the resistance will be enormously increased, the only escape of electricity being by the insulating supports, or through the gutta-percha sheath which surrounds the wire. In this case it is clear that the resistance is inversely proportional to the length of cable tested, and we shall have the proportion,

Resistance after fault : Resistance with distant end insulated
:: length of line : distance of fault,

whence the distance of the fault is found.

191. 2*nd Method.* When the wire is severed and the broken end makes complete earth.

Here the resistance will be diminished, since the electricity escapes to earth at the fault, instead of at the further end. Hence the resistance will be directly proportional to the length, and we have

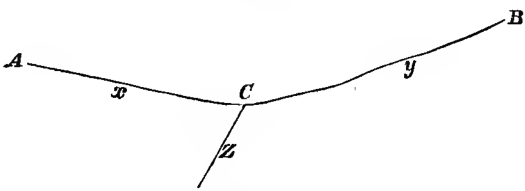
$$\left.\begin{array}{c}\text{Resistance of whole line with}\\ \text{end to earth}\end{array}\right\} : \text{Resistance of faulty line}$$
$$:: \text{length of line} : \text{distance of fault},$$

whence again the position of the fault can be found.

192. 3*rd Method.* When the wire is not completely broken, but makes partial earth. *Blavier's Formula.*

Let R be the resistance of the line AB when perfect,

Fig. 61.



S the resistance of the faulty line measured from A when B is to earth, T the resistance of the faulty line when B is insulated,

If x, y, z be the resistances of the portions AC, CB, and of the fault at C, we have

$$R = x + y \quad\text{..........................(i)},$$

$$S = x + \frac{yz}{y+z} \quad\text{..........................(ii)},$$

$$T = x + z \quad\text{..........................(iii)},$$

three equations for x, y, z.

By (i), $\qquad y = R - x$,

and by (iii), $\qquad z = T - x$.

Substituting in (ii),

$$S = x + \frac{(R-x)(T-x)}{R+T-2x},$$

$$(R+T)S - 2xS = (R+T)x - 2x^2 + RT - (R+T)x + x^2;$$

$$\therefore\ x^2 - 2xS = RT - (R+T)S;$$

$$\therefore\ (x-S)^2 = RT - (R+T)S + S^2$$

$$= (R-S)(T-S);$$

$$\therefore x = S - \sqrt{(R-S)(T-S)},$$

which is Blavier's formula.

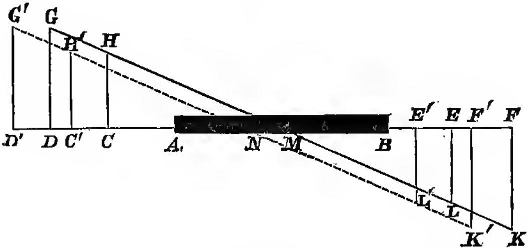
193. These methods all have the imperfection of assuming that the resistance at the fault remains constant (or vanishes) during the measurements, and of neglecting the leakage through the insulating sheath or supports. This, owing to polarization and a variety of irregularities, is not found to be the case. To get rid of this difficulty, various methods have been devised depending on the measurement of potential instead of resistance. The following, due to Dr Siemens, seems free from objection.

4th Method. Siemens' Method for a submarine cable or land-line.

Let AB represent the faultless cable insulated at both ends, AC, BE equal but variable resistances, CD, EF constant equal resistances. At D the positive pole of a battery is attached, and at E the negative pole of an equal battery, the opposite poles being to earth.

If DG, FK be the potentials at D and F, the line of fall of potential will cut AB in the middle, or the middle of

Fig. 62.



the cable will be at zero-potential. The *equal* differences $DG - CH$ and $FK - EL$ can be measured at the two ends by quadrant electrometers (for instance). If now a fault arise at N, the potential at this point some time after the attachment of the batteries will come to zero, so that the differences $DG - CH$ and $FK - EL$ will no longer be the same as before, or equal. If however we now alter the variable resistances increasing AC by CC', and diminishing BE by EE' when $CC' = EE'$, it is clear that by properly choosing these resistances we shall get the new line of fall of potential parallel to the old one, and passing through N instead of M. We shall then have $NC' = MC$; $ND' = MD$; $NE' = ME$; $NF' = MF$, and we shall have the same differences of potential at the two ends as before; in fact

$$G'D' - H'C' = F'K' - E'L' = DG - CH.$$

In this case the amount of resistances added at A and subtracted at B gives the distance of the fault from the middle of the cable towards A.

If the fault be at the middle of the cable, it is clear that the result is not affected by *normal* leakage, and if it be not at the middle, allowance can easily be made for the error it produces.

194. *5th Method.* When the core of a submarine cable is broken, while the sheath remains unbroken.

The best means in this case is to measure the electrostatic capacity of the broken part of the cable; then, knowing by

previous experiment the capacity per mile, a simple division gives us the distance of the fault.

A large number of other methods are used in practice, but the principles of them all will be understood from the foregoing examples. For descriptions of these methods the reader is referred to Mr Latimer Clark's *Electrical Measurements.*

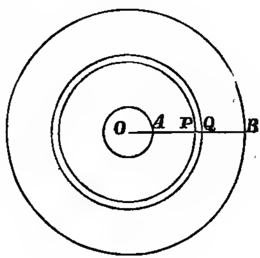
195. In our previous examples we have assumed the conductors cylindrical, and the lines of flow everywhere parallel to their length. These are the cases which most frequently occur in practice. We shall however give now two or three examples of calculating the resistance in conductors not linear in form.

Prop. VI. To calculate the resistance of a conductor bounded by two coaxial cylindrical surfaces. This will apply to the liquid in the circular form of Daniell's or Bunsen's cell.

Neglecting a portion near the ends, we shall assume the tubes of flow everywhere perpendicular to the axis of the cylinder.

Let the figure represent a section of the cylinder, and conceive it made up of concentric thin cylinders, such as PQ. The tubes of flow will be everywhere perpendicular to this cylindrical shell, and we may assume its resistance the same as for a linear conductor, whose section is its area, and length its thickness. Hence the resistance of the elementary cylinder

Fig. 63.



$$=\frac{1}{c}\cdot\frac{PQ}{2\pi\,.\,OP\,.\,l},$$

where c is specific conductivity, and l the length of the cylinder.

The resistance of the whole cylinder will therefore be

$$\frac{1}{2\pi cl}\cdot\Sigma\frac{PQ}{OP}.$$

But since, Art. 31*,

$$\frac{PQ}{OP} = \log\left(1 + \frac{PQ}{OP}\right) = \log\frac{OQ}{OP},$$

Resistance required

$$= \frac{1}{2\pi cl}\,\Sigma\,(\log OQ - \log OP).$$

Summing all successive differences, we have

$$\frac{1}{2\pi cl}(\log OB - \log OA)$$

$$= \frac{\log\dfrac{r_2}{r_1}}{2\pi cl},$$

where r_2 is the external and r_1 the internal radius of the cylinder.

This result is a particular example of the theorem of Art. 178, and might have been deduced from the capacity of the cylinder investigated Art. 118.

196. Prop. VII. To investigate the resistance of a large solid body of any form having two electrodes connected with the poles of a battery sunk in it to a considerable depth.

This investigation of course applies to the resistance of the Earth treated as the return line in Telegraphy.

We shall represent the two electrodes as two conductors sunk in the body, and charged with equal amounts of electricity, one positive and the other negative, and on the principles enunciated in Chap. III. we shall proceed to determine the form of the equipotential surfaces and lines of force.

The form of the electrodes will be of small importance except very near them, and we shall for convenience assume them to be spheres charged with quantities $+m$ and $-m$ of electricity.

In investigating the lines of force we must in theory take account of the distribution on the surface of the body. This produces a great complication in the theory, and we have here

taken the case in which the electrodes are deeply sunk, that we may neglect this distribution; in fact, the currents near the surface will be so weak that we may neglect them entirely.

On these suppositions the potential at any point distant r_1, r_2 from the electrodes will be

$$m\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

If the mass, to begin with, be at zero-potential, and $\pm V$ the potentials at the two electrodes whose radii we will take to be ρ, then $m=\rho V$, and the whole electromotive force between the electrodes is $2V$.

To find the intensity of the current, we have to consider that the flow across an equipotential surface of area σ is $cF\sigma$, where F is the resultant force, and c the specific conductivity: then the whole current-strength will be given by

$$I=c\Sigma F\sigma,$$

when the summation extends over any equipotential surface.

Now the electrodes are themselves equipotential surfaces, and we may consider the summation to take place over either electrode.

Since r_2 may be regarded as infinitely large (compared to ρ) for any point on the surface of the positive electrode, the potential over this electrode is $\dfrac{m}{\rho}$.

And on the same hypothesis the force will be $\dfrac{m}{\rho^2}$.

Also the area of the electrode $=4\pi\rho^2$;

$$\therefore\ \Sigma F\sigma=4\pi\rho^2.\frac{m}{\rho^2}=4\pi m=4\pi\rho V,$$

or if E be the whole electromotive force between the two electrodes $E=2V$, and we have

$$I=2\pi c\rho E.$$

But if R be the whole resistance,

$$I=\frac{E}{R}.$$

$$\therefore R = \frac{1}{2\pi\rho \cdot c}.$$

Hence we see that the resistance is independent of the distance between the electrodes but varies inversely as their linear dimensions.

197. It may be interesting to notice that we might have deduced the same result by considering the plane which bisects the line joining the electrodes at right angles.

The potential at any point on this plane clearly vanishes since we have for the potential

$$m\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

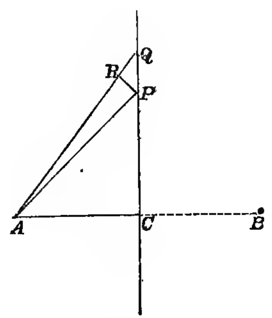
and $$r_1 = r_2.$$

The negative may then be regarded as the electrical image of the positive electrode, and the force at any point P on this plane is, Art. 104, represented by

$$\frac{2mp}{AP^3},$$

where m = quantity of electricity, p = perpendicular distance of the plane from the electrode.

Fig. 64.



For the annulus corresponding to PQ we have

$$F\sigma = \frac{2mp}{AP^3} \cdot 2\pi CP \cdot PQ.$$

But $CP \cdot PQ = AP \cdot QR$, where PR is perpendicular to AQ.

Hence
$$F\sigma = \frac{2mp}{AP^3} \cdot 2\pi AP \cdot QR$$
$$= 4mp\pi \frac{AQ - AP}{AP^2}$$
$$= 4mp\pi \cdot \frac{AQ - AP}{AP \cdot AQ}$$
$$= 4mp\pi \left(\frac{1}{AP} - \frac{1}{AQ}\right).$$

Hence for the current-strength across this surface
$$I = c\Sigma F\sigma = 4mpc\pi \left(\frac{1}{p}\right)$$
$$= 4mc\pi.$$

Exactly the same result as for the current across the electrode.

In this we have assumed no approximation except that the body under consideration is of infinite extent.

198. The result indicated in the last Article but one is found to agree fairly with experiments.

This investigation also shows the importance of placing the electrodes in moist ground whose specific resistance is small; otherwise the resistance offered by the first layers of the soil round the electrode may be much greater than that of the whole of the rest of the earth. When a badly conducting portion occurs at a distance from the electrodes, the principle of divided circuits shows us that it produces a very small effect indeed on the whole resistance.

199. Cor. The same method may be applied to the explanation of Nobili's rings when the fluid stratum is very thick compared to the distance of the electrode from the metal plate.

These rings, as is well known, are produced when a wire forming one electrode is immersed in a fluid spread over

a metal plate forming the other electrode, these electrodes being connected with a battery. If the fluid is an electrolyte it will be decomposed, and some product of decomposition will be laid in thin layers on the metal plate. The varying colour produced by thin plates according to their thickness will vary with the thickness of this deposit, and the thickness of the deposit will depend at each point on the intensity of the electrical force.

Now this will be *electrically* the problem we investigated (Art. 104) under electrical images when we had an electrified point placed near an infinite plate.

The force at each point on the plate will be that due to the electrified point combined with its electrical image, and is shown to vary inversely as the cube of the distance between the electrode and the point on the plate. Hence the thickness of the deposit will vary also as the inverse cube of the distance. The remaining part of the investigation having reference to the succession of the coloured rings produced, being an optical problem, is out of place here.

200. We have shown that the amount of electricity transmitted across any section of a tube of force is proportional to a quantity c which depends only on the nature of the substance. If the substance contained in any tube be not *isotropic*, c will vary, and we shall have different current-strengths in different parts. As a consequence we shall have a charge of electricity gradually developed at the surface bounding the heterotropic parts of the tube. We now give an example of this kind.

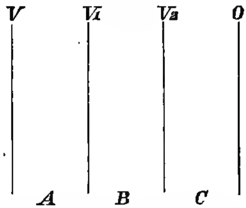
Prop. VIII. A stratified plate composed of parallel isotropic laminæ has its opposite faces kept at given potentials, to find the amount of the electric charge at the surface bounding two layers.

Let there be three layers A, B, C, and let t_1, R_1, C_1, ρ_1, K_1 be the thickness, resistance, capacity, specific resistance, and inductive capacity respectively of A, and let similar letters with suffixes 2, 3 denote those of B, C.

Let the potentials at the surfaces A, B; B, C be initially V_1 and V_2, and finally V_1' and V_2'.

Let also the potentials at the outer surfaces be V and O.

Fig. 65.



Initially, since there is no electrostatic charge on the surfaces A, B and B, C, we shall have

$$-C_1(V-V_1)+C_2(V_1-V_2)=0 \text{ at } A, B,$$

and

$$-C_2(V_1-V_2)+C_3V_2=0 \text{ at } B, C;$$

$$\therefore \frac{V-V_1}{\frac{1}{C_1}}=\frac{V_1-V_2}{\frac{1}{C_2}}=\frac{V_2}{\frac{1}{C_3}};$$

$$\therefore \text{ each of them} = \frac{V}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}=\frac{V}{\frac{1}{C}} \text{ suppose.}$$

These equations give

$$V-V_1=\frac{C}{C_1}V;\ V_1-V_2=\frac{C}{C_2}V;\ V_2=\frac{C}{C_3}V.$$

Next for the current-strength in A, B, C respectively, we have

$$I_1=\frac{V-V_1}{R_1},$$

$$I_2=\frac{V_1-V_2}{R_2},$$

$$I_3=\frac{V_2}{R_3}.$$

Hence $I_1=I_2=I_3$ only if

$$\frac{V-V_1}{R_1}=\frac{V_1-V_2}{R_2}=\frac{V_2-V_3}{R_3}=\frac{V}{R_1+R_2+R_3}=\frac{V}{R} \text{ suppose.}$$

Hence if the current in these plates is the same,

$$\frac{C}{C_1}V = \frac{R_1}{R}V, \text{ or } R_1C_1 = R_2C_2 = R_3C_3 = RC.$$

If this relation does not hold good, a gradual storing up of electricity will take place at the surfaces A, B and B, C. The law of the development of these charges is complicated, but their ultimate amount we can easily see. In fact, the storing up will go on until the currents in the three plates are equal. Hence if V_1' and V_2' be the ultimate potentials at A, B; B, C respectively,

$$\frac{V - V_1'}{R_1} = \frac{V_1' - V_2'}{R_2} = \frac{V_2'}{R_3}, \text{ and therefore} = \frac{V}{R}.$$

Under these conditions the charge bound on the plate A at the surface A, B

$$= - C_1(V - V_1');$$

that bound on plate B at surface A, B

$$= + C_2(V_1' - V_2');$$

and that bound on the plate B at surface B, C

$$= - C_2(V_1' - V_2');$$

and that bound on plate C at surface B, C

$$= + C_3V_2'.$$

Hence the whole charge on the surface A, B

$$= - C_1(V - V_1') + C_2(V_1' - V_2')$$

$$= -\frac{R_1C_1}{R}V + \frac{R_2C_2}{R}V = \frac{V}{R}(-R_1C_1 + R_2C_2)$$

and the charge on the surface B, C

$$= - C_2(V_1' - V_2') + C_3V_2'$$

$$= \frac{V}{R}(-R_2C_2 + R_3C_3).$$

But by Art. 178,

$$R_1C_1 = \frac{1}{4\pi}\rho_1K_1; \quad R_2C_2 = \frac{1}{4\pi}\rho_2K_2; \text{ and } R_3C_3 = \frac{1}{4\pi}\rho_3K_3.$$

Hence the amount at surface A, B

$$= \frac{V}{4\pi R}(-\rho_1 K_1 + \rho_2 K_2),$$

and the amount at surface B, C

$$= \frac{V}{4\pi R}(-\rho_2 K_2 + \rho_3 K_3).$$

The same theory might be extended to any number of plates.

It is to be noticed in the above results that the charges accumulated at the surfaces are independent of the thicknesses of the plates, and depend only on the change in value of ρK.

If the outer surfaces be brought to zero, it is clear that this accumulation within the conductor will in part be conducted back again in reverse order to the exterior, until the whole is discharged.

COR. If only some of the plates be conducting and others non-conductors, the same general effects will follow. Thus if B be a conductor, charges will be developed on the surfaces A, B and B, C, which will be bound across the dielectric to parts of the charges on A and C, which parts will by that means be disguised. The amounts so disguised the student can easily investigate for himself.

201. This proposition is important, since in the opinion of Prof. Clerk Maxwell and M. Gaugain it is the best explanation yet offered of the phenomenon of 'electrical absorption,' as observed in the residual charge of a Leyden jar. Faraday attributed it to a partial soaking of the electricity from opposite sides of the dielectric into its substance. This explanation contradicts Faraday's own principle of the impossibility of charging any mass of matter bodily with electricity. The explanation furnished by the stratified medium shows that the accumulation takes place after the manner of a compound condenser on strictly electrostatic principles, each charge being bound across the dielectric to an equal amount of opposite electricity. As a minor point this explanation confirms the observed fact that with air as dielectric there is no residual charge. We can easily see, however, that whatever substance except air the dielectric be composed of,

owing to imperfect annealing during cooling, and the necessary irregularity in composition, the medium will not be perfectly isotropic, and where the medium is not isotropic the phenomenon of internal accumulation must arise. Prof. Clerk Maxwell justly remarks, that although this theory shows a possible way in which the residual charge may arise it is conceivable that it really points to an entirely new kind of polarization in the dielectric.

Examples on Chapter VII.

The following is a table of conductivity of the commonest metals:

Silver	100.	Copper	99·9	Zinc	29.
Platinum	18.	Iron	16·8	German Silver	7·7
Mercury	1·6	Graphite	0·07.		

The following results of experiment may be assumed:

Resistance per statute mile of pure copper wire $= \frac{54892}{d^2}$ ohms, where d = diameter in thousandths of an inch.

Resistance of 1 knot (2029 yds.) of pure copper wire weighing 1 lb. at 32° F. = 1091·22 ohms. (Latimer Clark.)

1. Find the length of copper wire $\frac{1}{16}$ in. in diameter whose resistance is one ohm. *Ans.* $125\frac{1}{4}$ yds.

2. Find the length of platinum wire, of the same diameter as question 1, which has one ohm resistance.
Ans. $22\frac{1}{2}$ yds.

3. Siemens' unit is defined to be the resistance of a column of mercury one metre long, whose section is a square mm. Compare it with the ohm.
Ans. Siemens' unit = 1·08 ohm.

4. Find the resistance of a fuse of platinum wire $\frac{3}{8}$ in. long, of which a yard weighs 2 grains. Given that the specific gravity of platinum is 22, and of copper 8·8.
Ans. ·133 ohm.

5. Find in ohms the resistance of a hundred miles of iron telegraph wire, whose diameter is $\frac{1}{5}$ in. *Ans.* 816 ohms.

6. Find the diameter of a copper wire of which one mile gives 738 ohms resistance. *Ans.* ·0086.

7. What must be the ratio between the diameters of a copper and iron wire that equal lengths may give the same resistance? *Ans.* 41 to 100 nearly.

8. What length of german-silver wire ·05 in. diameter must be taken to get one ohm resistance? *Ans.* 6·178 yds.

9. What diameter must a silver wire have that one metre may have one ohm resistance? *Ans.* ·0058in.

10. Resistance-coils are made of german-silver wire, ·005 in diameter. Find the length of it in a coil whose resistance is 1000 ohms. *Ans.* 61·783 yds.

11. If 100 in. of copper wire weighing 100 grs. has resistance ·1516 ohm, find the resistance of 50 in. weighing 200 grs. *Ans.* ·01895 ohm.

12. Two cells, each one ohm internal resistance, are connected in compound series with a wire whose resistance is one ohm. If each of these, when connected singly by stout wires to a galvanometer * of no appreciable resistance, deflect it 25°, how much will the combination deflect it? *Ans.* 17° 15′.

13. A single thermo-electric couple deflects a galvanometer of 100 ohms resistance 30′, how much will a hundred such couples in compound series deflect it? (The resistance of the couples themselves may be neglected.) *Ans.* 41° 7′.

14. The internal resistance of a cell is half an ohm; when a galvanometer of one ohm resistance is connected with it by short thick wires it is deflected 15°: by how much will it be deflected if for one of the thick wires a wire of 1·5 ohms resistance be substituted? *Ans.* $7\frac{2}{3}°$.

15. A cell of $\frac{1}{3}$ ohm resistance deflects a galvanometer of unknown resistance 45°, the connection being made by short stout wires. If a wire of 3 ohms resistance be substituted for one of the stout wires the deflection is 30°. Find the resistance of the galvanometer. *Ans.* 3·8 ohms nearly.

* The galvanometer, unless the contrary be stated, may be assumed to be a tangent galvanometer.

16. A galvanometer of no sensible resistance is deflected 45° by a cell connected with stout wires. When a resistance of 5 ohms is introduced, the deflection sinks to 30°; find the resistance of the cell. *Ans.* 6·8 ohms.

17. A Bunsen and a Daniell cell are placed in the same circuit, first with their electromotive forces in the same direction, and secondly in opposite directions, the deflections being respectively $30^\circ{\cdot}2$ and $10^\circ{\cdot}6$. Compare their electromotive forces. *Ans.* Bunsen's cell = 1·9 of Daniell's.

18. If an insulated closed voltaic circuit be connected at a point whose potential is V with an insulated conductor whose capacity is C', show that the potential at this point becomes $\dfrac{CV}{C+C'}$, where C is the capacity of the original circuit; and that there will be a fall of potential through the whole circuit equal to $\dfrac{C'V}{C+C'}$.

19. The terminals of an insulated battery of 520 cells are united by 78 miles of cable which have the same resistance as the battery. At the extremity of the cable, next the zincode, 43 miles of cable are connected with the circuit at one extremity, the whole being insulated. Show that the potential at the zincode immediately falls in the ratio 14 to 9 nearly.

20. A battery of seven cells has its ends joined by a wire whose resistance is three times that of the battery. At the junction of the third and fourth cells there is connection with the earth. Draw a diagram of the fall of potential in the circuit. How will the current in the circuit be affected?

21. Three Daniell's cells are arranged in compound circuit with a resistance of ten times one cell between each two. Calculate the current when the terminals are joined by a stout wire, and draw a diagram of the fall in potential through the circuit.

22. If the difference of potential between the two terminals of a battery be measured when the circuit is open, and if the same difference of potential, measured when closed, is one-half its former value, show that the external and internal resistances are equal.

23. If by introducing into a circuit (formerly closed by a short stout wire) a certain measured resistance, the current-strength sink to one-half its former value, show that the resistance introduced is equal to the internal resistance.

24. Find the resistance introduced into a circuit by using a galvanometer with a shunt, the resistances of the galvanometer and shunt being known.

25. Find the resistances in a series of shunts which shall allow $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, &c. of the current to pass through a galvanometer, the resistance of the galvanometer being known.

26. To a telegraph wire is attached at two points, five yards from each other, a wire of $\frac{1}{10}$th its diameter, but 20 yds. long, which is passed round a telegraph needle; what proportion of the current passes in the long and short wire respectively?

27. A line joining two places A, B, 130 miles apart, at 10 miles from A drops from its support and rests on another wire which makes earth at distances 30 and 40 miles. Find the ratio of the current-strengths at A and B.

Ans. If sent from A, 8 to 1, if from B, 12 to 19.

28. In a closed circuit, two points are joined by a conductor of given resistance. Given the resistances, write down equations to determine the currents in all the branches.

29. Two cells AA_1, BB_1, are simple circuited by wires ACA_1, BDB_1, and the points C, D joined by a wire. Given the resistances and electromotive forces, find the current in CD.

30. Find the current in the preceding question, supposing the two cells arranged in a compound circuit.

31. Show in the last question if the current in CD vanish, and E_1, E_2 be the electromotive forces in AA_1, BB_1 respectively, then

$$E_1 : E_2 :: \text{resistance in } CAA_1D : \text{resistance in } CBB_1D.$$

32. Four cells, the resistance in each being 3 ohms, are used with external resistance 10 ohms; will it be better to use them in a simple or compound circuit?

Ans. Compound circuit.

33. Find the smallest external resistance in the preceding question with which it will be an advantage to use a compound circuit. *Ans.* 3 ohms.

34. When the poles of 100 cells in compound circuit are joined by a thick wire, a galvanometer deflects 60°; when 100 ohms are inserted the deflection sinks to 30°. Find the internal resistance of one cell. *Ans.* 5 ohms.

35. Of two cells one is short circuited and gives 60° deflection, and on introducing 6 ohms the deflection becomes 45°: another, when short circuited, gives 45°, and on introducing 6 ohms sinks to 30. Find the ratio of their electromotive forces. *Ans.* $\sqrt{3}$ to 1.

36. The zincode of a battery of 100 cells is to earth, and the other end communicates with the end A of a line AB, whose distant end B is to earth. The resistance of the line AB is ten times that of the battery. If now a second battery of 50 cells having also its zincode to earth have its other end (as well as that of the former battery) to the end A of the telegraph, find the change in the current in the line.

Ans. Current is $\frac{22}{31}$ of former value.

37. A battery of 20 ohms resistance sends a current through a galvanometer of 15 ohms resistance to a line of 70 ohms resistance, and at the other end is a galvanometer of 15 ohms resistance. What effect is produced on each galvanometer if there be a fault whose resistance is 20 ohms in the middle of the line.

Ans. Current in battery galvanometer is altered in ratio 84 to 59, that in line galvanometer in ratio 24 to 59.

38. If the position of the fault were unknown, show how it might be inferred from the readings of the galvanometers.

39. In a compound arrangement with 3 cells and no external resistance except a galvanometer the deflection was observed to be 60°. Using one cell only and the same external conditions the deflection was 44°. On introducing into the latter arrangement 20 ohms additional resistance the

deflection sank to 25°. Find the resistance of the galvanometer and of each cell of the battery.

Ans. Internal resistance 8 ohms.
Galvanometer „ 7 ohms.

40. One hundred cells each of internal resistance 4 ohms are to be used with 25 ohms external resistance. Find the arrangement which will give the strongest current and the strength of this current.

Ans. 4 rows of 25 cells. Current-strength $\frac{1}{2}E$.

41. What is the best arrangement of 6 cells each of $\frac{2}{3}$ ohm resistance against an external resistance of 2 ohms?

Ans. 6 cells in compound circuit, or 2 rows of 3 cells.

42. What is the best arrangement of 20 cells each of 8 ohms resistance against an external resistance of 4 ohms?

Ans. 2 rows of 10 cells.

43. A battery of three cells is arranged in a mixed circuit so that there are two rows containing 1 and 2 cells respectively, and the terminals are connected by a wire of resistance R. Find the current-strength.

Ans. $\dfrac{4E}{3r+2R}$, where E is the electromotive force and r the resistance in each cell.

44. A battery of six cells is arranged in mixed circuit so that there are three rows containing respectively 1, 2 and 3 cells. Find the current-strength in a conductor joining the terminals.

Ans. $\dfrac{18E}{6R+11r}$.

45. A battery is arranged in mixed circuit consisting of n rows containing respectively 1, 2, 3......n cells. Show that the current-strength in a wire joining the terminals is given by

$$\frac{nE}{R+rS},$$

where $$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots + \frac{1}{n}.$$

46. In Wheatstone's bridge as commonly used (Fig. 57), show that when the branch BD is open the difference of potential between B and D is given by

$$\frac{E(ps - rq)}{R(p+q+r+s)+(p+q)(r+s)}.$$

47. Find also the electromotive force in BD when this branch is closed.

48. If $p = 16$, $q = 4$, $r = 4{\cdot}1$, $s = 1{\cdot}03$ and $R = 4$, and the galvanometer resistance is very great, find the part of the whole electromotive force which acts in the bridge-piece.

Ans. $\dfrac{1}{2539}$.

49. A polarized voltameter and a galvanometer are included in the bridge of Wheatstone's bridge and the resistances arranged so that there is no current in the bridge; show how to determine the electromotive force of polarization.

50. A voltameter or polarizable cell is included with a commutator in the branch AB suppose of the bridge, show what measurements you would make to determine the effect (if any) produced by the polarization in the resistance.

51. If the resistances in the preceding question were so arranged that no current was passing in the bridge BD, and if on turning the commutator no current still was passing, what inference would you draw?

52. ABC is a triangle formed by straight and uniform conductors, and OA, OB, OC are similar conductors joining O to the angular points; find the condition that OA may be conjugate to BC.

53. Show that in the preceding question if O be a point such that OA and BC are always conjugate, the locus of O is a circle.

54. Show also there is one and only one point such that OA is conjugate to BC, OC to AB and OB to AC.

55. In a wire joining the poles of a galvanic cell of small resistance the wire is more heated if it be of copper than if it be of platinum of the same dimensions, but if the internal resistance be large, the platinum wire will be more heated than the copper. Explain this, and reconcile it with the statement that the heat evolved is equal to the energy which runs down in the discharge.

56. Show in the previous question that the heat given out in the two wires would be equal, if the internal resistance of the cell be a geometric mean between the resistances of the copper and platinum wires.

57. If a chain made of alternate links of platinum and silver have a strong current sent through it the platinum becomes red hot while the silver remains cold and black. Explain this.

58. If part of a loop of wire which is rendered red hot by the passage of a current be held in a spirit flame, the part outside the flame becomes black, but if the same part be dipped in water the part outside immediately glows with a redder light. Explain this.

59. A copper wire carrying a voltaic circuit is dipped in copper sulphate or dilute acid, describe the effect produced on the current and on the liquid.

60. In a zinc-copper cell the zinc and copper plates are united by a copper wire immersed in the liquid, what will be the effect on the current in the circuit?

61. A series of plates of platinum foil separated by paper damped with acidulated water are connected for a few minutes with a battery. After removal the terminals are united with a galvanometer, which immediately indicates a current. Explain this, and show the direction of the current in relation to the battery current. Will this current be permanent? How far may the arrangement here described be compared to a Leyden battery?

CHAPTER VIII.

MAGNETISM.

202. We have here a new class of phenomena to consider. They are exhibited most strongly in the varieties of iron, though probably in some degree in almost all bodies. We shall place before the reader a brief sketch of the phenomena with which we assume he is already familiar, and develop as far as possible step by step the theory deduced from them.

203. *Experiment* 1. A piece of magnetic iron or a bar of steel magnetized is found to exercise a peculiar force on pieces of iron. This force vanishes near the middle of the bar, and increases in magnitude very rapidly towards the ends. The force is often spoken of as resident in the ends of the magnet, which are therefore called its *poles*.

As in electricity the force is often attributed to an imaginary distribution of *magnetic fluid* over the poles. This must of course be treated as only an image representing to our minds the existing force.

204. *Experiment* 2. If the two poles of any magnet A be brought in succession into the neighbourhood of a pole of a second magnet B, one will suffer an attractive and the other a repulsive force. If another magnet C be taken, and the poles of A and C, which are both attracted or both repelled by one pole of B, be made to act on each other, there will be a repulsive force between them. If again two poles be taken, one of which is attracted and the other repelled by either pole of B, there will be an attractive force.

205. We infer from these experiments that *every magnet has two dissimilar poles, and that like poles repel each other, but unlike poles attract each other.*

The nomenclature of these poles is derived from the behaviour of the magnet when suspended about its centre of gravity. Each magnet, if free from other magnetic influence, then points in this country nearly due N. and S. The end that points northwards is called the north pole, and that which points southwards the south pole.

We shall however speak of the magnetism distributed over the north pole of a magnet as *positive*, and that over the south pole as *negative* magnetism.

206. *Experiment* 3. The forces exerted by the two poles of a magnet on any third pole are always equal in magnitude, though opposite in direction. This will be conveniently expressed by saying that if the strength of one pole be $+m$, that of the other is $-m$.

The strength of a pole can be expressed by the force it exerts on another pole, and the unit in terms of which it is measured can be defined thus:

DEF. A UNIT MAGNETIC POLE *is a pole which exerts a unit of force (or a dyne) at unit distance on another equal pole.*

As in statical electricity, we here define Magnetic density.

DEF. MAGNETIC DENSITY *at any point on the surface of a magnetised mass is the quantity of magnetism per sq. cm. of surface separated at that point.*

207. *Experiment* 4. The force between two magnetic poles is found to vary as the product of their strengths when at the same distance, and inversely as the square of their distance when the distances are varied.

Thus if we have two poles whose intensities are m and m' at a distance r from each other, the force between them is $\frac{mm'}{r^2}$, this force being repulsive when the numerator is positive, and attractive when the numerator is negative. Comparing this with the statement of the law of gravitation in Art. 30, we infer that we may apply to magnetism all the propositions of Chap. II. relating to matter. We can assume therefore the following propositions.

I. If there be a number of magnetic poles whose in-

tensities are m_1, m_2, m_3...... at distances r_1, r_2, r_3...... respectively from a given point, the potential at this point is represented by

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \ldots\ldots\ldots \text{ or } \Sigma \frac{m}{r},$$

which represents the work done on or by a unit pole carried from this point to an infinite distance.

Assuming the unit pole always positive, the work will be done *on* the pole when $\Sigma \frac{m}{r}$ is negative, and *by* the pole when $\Sigma \frac{m}{r}$ is positive (see Art. 62).

II. The force at any point in any direction will equal the rate of change of potential in that direction (see Art. 37).

III. There will exist for a magnetic distribution a system of surfaces over which $\Sigma \frac{m}{r}$ is constant, which may be called equipotential surfaces; and there will be a system of curves cutting them at right angles called lines of force, such that the direction of the resultant force at that point will be a tangent to the line of force through the point (see Arts. 38—41).

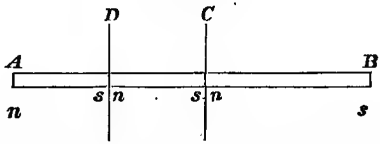
IV. By means of the lines of force the magnetic field may be mapped out into tubes of force through each of which $\Sigma F\sigma$ is constant, supposing no magnetism to exist within the tube. If such should exist, $\Sigma F\sigma$ changes by $4\pi m$, where m is the amount of magnetism cut through, on passing from one side to the other of such magnetism (see Arts. 47, 48).

A common way of expressing the first part of this proposition is by assuming any equipotential surface so mapped out that the number of lines of force arising from a unit of area anywhere on the surface is always numerically equal to the average force over that area. The proposition shows that with this hypothesis the number of lines of force cutting through any small plane area whatever is equal to the force resolved perpendicular to this area multiplied by the area.

This is manifest when the area forms part of an equipotential surface. If the small area σ be inclined to the surface at an angle θ, its projection on the surface will be $\sigma \cos \theta$, and the number of lines of force through it will be $F\sigma \cos \theta$, which is at the rate of $F \cos \theta$ per unit of area. But $F \cos \theta$ is the force resolved perpendicular to σ. Hence the proposition in its most general form.

208. *Experiment* 5. If a straight bar magnet AB, such as was referred to in preceding experiments, be broken in

Fig. 66.



half as at C, we do not get two bars, one AC charged entirely with north magnetic fluid, and BC entirely with south magnetic fluid; but at C two new poles are developed on opposite sides of the plane of division, so that we get two magnets with poles of the same intensity as the original magnet.

If AC be broken at D, we find again two new poles developed at the place of fracture, and this operation may be repeated indefinitely, each fragment broken off still being magnetized similarly to the given magnet.

This leads us to the conception of a magnet as made up of molecules, each of which is a magnet, the resultant magnet being due to the combined action of all the elementary magnets of which it is composed.

This also shows us that in any magnetized mass the amount of positive and negative magnetic fluid is always the same, nothing in magnetism corresponding to conduction ever having been observed whereby the magnetism can flow from molecule to molecule.

209. *Experiment* 6. A magnetic pole induces in a piece of soft iron near it a separation of the magnetic fluids; on the parts nearest to it inducing a distribution of magnetism of opposite sign, and on the parts more remote a distribution of magnetism of like sign with itself.

210. *First.* Let the magnetizable mass be a small filament of iron, whose length is along a line of force. We may regard it as a small portion of a tube of force. The separation of the magnetic fluids across any equipotential surface is the same, and will be jointly proportional to the magnetic force, and to the area of the section. The intensity of the free magnetism on the ends of the filament will therefore depend

(i) On the nature of the iron,

(ii) On the intensity of the force.

(i) That depending on the nature of the iron may be called the coefficient of magnetic induction, and we will define it thus.

DEF. THE COEFFICIENT OF MAGNETIC INDUCTION *for a given iron is the amount of magnetism separated on a prism of the iron whose section is a unit area placed along lines of force in a field of unit intensity.*

This coefficient we call k, and it will be greatest in soft iron, least in steel.

(ii) As far as the force is concerned, it is clear that if the section of the filament be α, F the force, and k the coefficient of magnetic induction, the amount of magnetism separated at the two ends of the small filament will be $\pm k . F . \alpha$.

After removal from the field of force, the filament generally retains some portion of the magnetism induced in it. This is explained by assuming a certain coercitive force in the iron which depends on the nature of the iron, being greatest in steel and least in soft iron. Where the coercitive force is great the amount of magnetism retained may be largely increased by putting the bar in a state of vibration, its molecules being thus enabled more easily to take up a new magnetic condition under the influence of the field.

211. *Secondly.* Let the mass be of any form placed anywhere in a magnetic field. Here the amount of magnetic separation in each element becomes very complicated. To find it we must conceive the element removed, and compute the resultant force at any point within the cavity so produced. This resultant will consist of two parts:

(i) That due to external bodies or the ordinary magnetic force,

(ii) That due to the magnetism distributed through the whole mass of the body.

212. We may here point out the analogy between a magnetic mass subject to magnetic force, and a dielectric subject to electrical force. In both we have molecules in which the electric or magnetic fluids are separated, only to complete the analogy we must suppose the magnetic insulating material absolutely impervious; in both we have lines and tubes of force, the quantity of fluid separated across any equipotential surface in the tube being in the dielectric $\pm \frac{1}{4\pi} F\sigma$, and in the magnetic mass $\pm kF\sigma$, where k depends on the nature of the magnetic medium.

This analogy helps us to show that we may compute the whole effect of the second component in the last article by that of a distribution of magnetism over the surface of the mass. For we may replace the magnetic by an electrical distribution of the same numerical intensity at each point, and Art. 72 shows that this may be replaced by a superficial distribution.

213. Two cases of magnetization are worth special notice.

Prop. I. A. To investigate the magnetization of a straight cylindrical bar placed in a uniform magnetic field with its length parallel to the lines of force.

In this case the whole bar is a tube of force, and the intensity of magnetic separation is the same everywhere along it. We may conceive it made up of rows of molecules in which the magnetic separation is represented thus.

Fig. 67.

The potential at an external point O is represented by

$$m\left(\frac{1}{OA}-\frac{1}{OB}+\frac{1}{OB}-\frac{1}{OC}+\frac{1}{OC}-\frac{1}{OD}+\ldots\ldots-\frac{1}{OP}\right),$$

where m is the amount of fluid separated in each molecule. Hence the potential of this row of molecules is

$$m\left(\frac{1}{OA}-\frac{1}{OP}\right)=m\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \text{ suppose.}$$

The same will be true for all the rows of molecules, and we have for the potential of the bar on any external point 0

$$\Sigma m\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

If the bar be long and very thin, r_1, r_2 will be sensibly constant over the respective ends, and we have

$$V=M\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

where M is the strength of the pole.

This is usually called a *solenoidal* distribution of magnetism, and is approached practically in the bar magnet. In all bar magnets there is a certain small force at a distance from the poles which is traceable to the falling off in strength of the magnetic separation in the molecules as we go towards the ends.

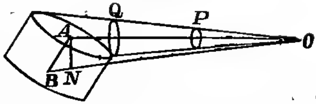
Cor. If a solenoidal magnet be bent round so that it forms any closed curve, the potential on any external point vanishes; for then we have $r_1=r_2$, and the expression above is zero.

214. Prop. II. A. To find the potential of a thin magnetic shell in which the magnetization is everywhere normal to its surface. This is called a lamellar distribution.

Such a shell may be conceived as made up of an infinite number of thin short magnets, placed side by side. Let the figure represent such an element, AB being the normal, σ the area of each pole, A being the north and B the south pole of the small magnet.

Join OA and OB, and draw AN perpendicular to OB. Also describe a cone whose vertex is O and base A, and conceive two sections of this cone, one P by a sphere of radius unity, and the other Q* by a sphere of radius OA.

Fig. 68.



We will suppose the intensity of the pole at O to be m, and the density of the magnetic fluid on A and B to be $\pm\rho$, the strength of each pole will then be $\pm\rho\sigma$.

The potential of the shell on the magnetic pole at O will then be

$$m\rho\sigma\left(\frac{1}{OA}-\frac{1}{OB}\right)=m\rho\sigma\,\frac{OB-OA}{OA\,.\,OB}=m\rho\sigma\,\frac{BN}{OA^2}$$

$$=\frac{m\rho\sigma\,.\,AB\cos\epsilon}{OA^2},$$

where ϵ is the angle ABO.

215. We may put this expression in a form which gives conveniently

(i) The potential of the shell-element on O,

(ii) The potential of the magnet pole at O on the shell-element.

(i) To find the potential of the shell on O. The magnet pole may here be assumed unity, and the potential on O becomes

$$\frac{\rho AB\,.\,\sigma\cos\epsilon}{OA^2}.$$

Now since the area σ is perpendicular to AB, and the area Q to OA, the angle between σ and Q must be ϵ, and Q may be regarded as the orthogonal projection of σ.

Hence $$Q=\sigma\cos\epsilon.$$

$\therefore$ the potential becomes

$$\frac{\rho\,.\,AB\times\text{area }Q}{OA^2}$$

* To prevent confusion Q is made a section near the edge of the shell, but in the reasoning the plane of Q is supposed to pass through A.

Again by similar figures,

$$\text{area } Q : \text{area } P :: OQ^2 : OP^2$$
$$:: OA^2 : 1,$$

since P is on a sphere of radius unity.

Hence the potential on $A = \rho AB \times \text{area } P$.

The area P is the projection by a cone of the edge of the elementary shell on a sphere of unit radius, and this may be defined as the spherical measure of the solid angle subtended at O by the shell-element. We will denote this solid angle by ω. If the product $\rho . AB$ is constant over the whole shell, of which AB is an element, the shell is said to be a simple magnetic shell, and this product is defined to be its strength. We shall denote it by the symbol j. The potential of the whole shell may then be written $\Sigma j\omega$, or if the magnetic shell be simple, this reduces to $j\Sigma\omega$, and the solid angles subtended by the elements can be simply added, and will give on summation the solid angle subtended by the whole shell.

Hence the potential of a simple magnetic shell of strength j on an external point will be $\pm j\Omega$, where j is the strength of the shell, Ω the solid angle subtended by its edge, the positive sign being given when O *faces* the positive surface of the shell.

216. (ii) To find the potential of a magnetic pole at O on the element AB of a magnetic shell.

The expression, Art. 214, can now be written

$$\rho AB . \frac{m}{OA^2} \cos \epsilon . \sigma.$$

The factor $\frac{m}{OA^2}$ is the force at A due to a pole m placed at O, and $\frac{m}{OA^2} \cos \epsilon$ is this same force resolved along AB, the normal to the shell.

Hence $\frac{m}{OA^2} \cos \epsilon$ is the number of lines of force per unit area passing through the shell-element: and $\frac{m}{OA^2} \cos \epsilon$

is the absolute number of lines of force due to m which pass through the shell-element.

Now any distribution of magnetism may be represented as a distribution of magnetic poles, and the above proposition be applied. The potential of any magnetic distribution becomes

$$\rho \, . \, AB \, . \, \sigma \Sigma \frac{m}{OA^2} \cos \epsilon,$$

and each term in the summation denotes the number of lines of force per unit area due to that element of the distribution, and therefore

$$\sigma \, . \, \Sigma \frac{m}{OA^2} \cos \epsilon$$

will be the number of lines of force due to the whole distribution, which pass through the element of the shell. If we call this number n, the potential of any magnetic distribution on AB may be represented by $\rho \, . \, AB \, . \, n$.

217. If the shell be a simple shell, $\rho \, . \, AB = j$ a constant, and the potential of any magnetic distribution on the magnetic shell is equal to $j\Sigma n = jN$, where N is the whole number of lines of force due to the given system intercepted by the shell.

To determine the sign of this potential we must make a convention concerning the lines of force.

DEF. POSITIVE AND NEGATIVE LINES OF FORCE. *The positive direction of a line of force is the direction in which a north or positive pole placed on it would tend to move.*

218. We can now complete the statement. The potential of any magnetic system on a magnetic shell is measured by $\pm jN$, where j is the strength of the shell, N the number of lines of force due to the given system enclosed by its edge, the positive sign being attached when the lines of force pass through the shell from the plus to the minus magnetic side.

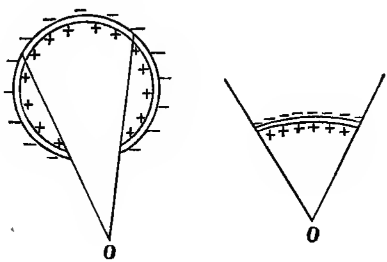
219. The foregoing proposition might be treated as a particular case of this, since the solid angle there defined is clearly the number of lines of force from a unit pole which would be intercepted by the shell.

The following six Articles, 220—225, which may be treated

as corollaries to the two propositions proved above, are of great importance.

220. Cor. 1. In estimating the potential of any magnetic shell on a point, we must remember that if some of the elementary cones cut the shell twice, the potentials of these elements will be equal and of opposite sign, and may therefore be neglected in the general summation. It is evident, on inspecting the figure, that in this case we have only to take account of the free edge, which will here be an *internal* edge.

Fig. 69.



221. Cor. 2. The potential of a closed shell on any internal point equals $\pm 4\pi j$, and on any external point vanishes.

The sum of all the solid angles subtended by elements of the shell at any internal point will clearly be the whole sphere, and this solid angle is measured by 4π, hence the potential is $\pm 4\pi j$. For any external point we notice that *all* the elementary cones cut the shell twice, and the whole potential therefore vanishes.

222. Cor. 3. In the case of a shell in the form of a plane lamina the potential anywhere on the positive side is $2\pi j$, and on the negative side $-2\pi j$, for the solid angle subtended by a plane at any point on the plane is clearly a hemisphere. We see also that the work done in bringing a unit pole from the negative round to the positive side of the shell is $4\pi j$, and is independent of the path taken.

223. Cor. 4. The potential of the plane lamina at any point in the plane of the shell outside the shell is zero. The solid angle subtended by the shell clearly in this case vanishes.

224. Cor. 5. Since the potential measures the work done on a unit pole, we see generally that if a magnetic pole of strength m be moved in the field of a given shell from a position in which the solid angle subtended is Ω_1 to a position in which it becomes Ω_2 the work done will be measured by

$$mj(\Omega_2 - \Omega_1).$$

If Ω_2 be less than Ω_1 the work done is negative, or the pole acquires kinetic energy during the movement.

More generally if a magnetic shell be moved about in the magnetic field from a position in which the number of lines of force enclosed is N_1 to another in which the number of lines becomes N_2, the work done in the movement is

$$j(N_2 - N_1);$$

here also the work done is negative if $N_2 < N_1$, or the force acting on the shell *helps* the motion.

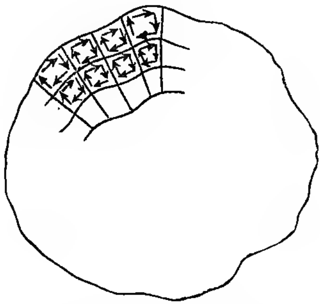
225. Cor. 6. It appears from the last result that a magnetic shell free to move about in a magnetic field will place itself in a position where its potential is as low as possible, or in the position which includes the greatest number of negative lines of force.

226. *Experiment* 7. A small closed voltaic circuit placed in the magnetic field is acted on by forces proportional to the forces which would be experienced by a thin magnetic shell whose edge coincides with the circuit, the strength of the current bearing a certain proportion to the strength of the shell, the direction of the current being such that an observer looking down on the north side of the shell sees the current following a direction opposite to the hands of a watch.

227. This experimental law can be extended to a voltaic circuit of any size and shape whatever. For con-

ceive the voltaic circuit filled up by a surface, and this surface divided into a number of closed curves by lines cutting each other at right angles, whose distances are small compared with the curvature of the surface. Conceive currents of the same intensity to circulate round each of these closed curves, as shown in the figure, all in the same direction.

Fig. 70.



Each closed curve may be regarded as a plane circuit, and for it by the above experiment may be substituted a small magnetic shell whose strength is in a certain ratio to the current-strength, and similarly for all the other elementary circuits; and the magnetic shell-elements substituted for each will all have the same strength.

But these shell-elements will make up a simple magnetic shell whose edge coincides with the original closed circuit.

Again, in the figure it is evident that along each side of the elementary closed circuit will be two currents of equal strength in opposite directions, which will therefore neutralize each other; the only parts not neutralized in this way being the elements which compose the original voltaic circuit.

228. Hence we see that as far as actions in the magnetic field are concerned we may substitute for any voltaic circuit a magnetic shell whose edge coincides with the circuit carrying the current, and whose strength bears a certain ratio to the current strength.

229. We may define the positive direction of the current in the circuit in the following way:

Def. Direction of Current. *The positive direction of the current is related to the positive direction of the lines of force in the same way as the direction of rotation to that of propulsion in a right-handed screw.*

This direction can be conveniently remembered by the twist in the muscles of the wrist in driving in a corkscrew. The opposite direction will be referred to as a left-handed screw, and the set of directions indicated above will be referred to as *right- and left-handed cyclical order.*

230. The experiment and deductions given above form the basis of the science of Electro-Magnetism.

It is usual here so to change our Electrostatic units that the circuit carrying a unit current, and a unit magnetic shell, shall be identical in their electromagnetic actions. Our former units of electromotive force, resistance, &c. will all have to be altered; but we shall assume at present that they are altered in such proportion that Ohm's formula remains unchanged, as also the formula for energy expended in the circuit.

The relations of the various units in the electrostatic and electromagnetic systems to each other we shall indicate in the next chapter.

231. We have now shown that the forces acting on a magnetic shell and on a voltaic circuit coinciding with its edge are identical, and since these forces are, in the case of the shell, derived from a magnetic potential, we shall assume that an identical electromagnetic potential exists from which the forces acting on the voltaic circuit are derived.

Before doing so, we must notice that the positive direction of a line of force passes from the positive to the negative side of a magnetic shell placed in the field (Art. 218), while the current in the equivalent circuit is left-handed or negative (Art. 226 and 229). We must remember therefore in applying propositions proved for magnetic shells to voltaic

circuits, that work done will be represented by potential with sign changed.

232. To keep this clearly before the student, we place here the conventions made and the properties proved for a magnetic shell, while in a parallel column we place the conventions made and properties deduced for a voltaic circuit.

MAGNETISM.	ELECTRO-MAGNETISM.
233. DEF. *The positive direction of a line of magnetic force is the direction in which a north pole placed on it tends to move.* Art. 217.	DEF. *The same.*
234. DEF. *The potential of a magnetic shell is positive when lines of force pass from positive to negative magnetism.* Art. 218.	DEF. *The potential of a voltaic circuit is positive when the lines of force and direction of current are related in right-handed cyclical order.* Art. 229.
	Experiment. The magnetic shell and voltaic circuit are equivalent when the lines of force and direction of current are in left-handed cyclical order.
	Deduction. The algebraical signs of the potential of a shell and its equivalent voltaic circuit will be opposite.
235. **Prop. I A. The potential of a magnetic shell in a magnetic field is measured by the product of its strength into the number of lines of magnetic force counted algebraically it encloses. Art. 217.**	**Prop. I. The potential of a voltaic circuit in a magnetic field is measured by the product of the current-strength into the number of lines of magnetic force counted algebraically enclosed by the circuit.**
236. **Prop. II A. The numerical value of the potential of a magnetic shell on a point is equal to the strength of the**	**Prop. II. The numerical value of the potential of a voltaic circuit on a point is equal to the product of the current-**

MAGNETISM.

shell multiplied by the solid angle subtended at the point by its edge. Art. 215.

237. **Prop. III A.** The potential of a plane magnetic shell of strength j is on one side $2\pi j$ and on the opposite $-2\pi j$, the difference $4\pi j$ representing the work done in bringing a unit pole round the edge from the negative to the positive side. Art. 222.

238. **Prop. IV A.** The work done on a unit pole in carrying it from a point A where the potential of a shell is $j\Omega_1$ to a point B where the potential is $j\Omega_2$ is

$$j(\Omega_2 - \Omega_1).$$

Art. 224.

ELECTRO-MAGNETISM.

strength multiplied by the solid angle subtended at the point by the circuit.

Prop. III. The potential of a plane voltaic circuit, carrying a current of strength i, is on one side $2\pi i$ and on the other $-2\pi i$, the difference $4\pi i$ representing the work done on a unit pole in bringing it round *outside the circuit* from the negative to the positive side.

COR. Since in the case of a voltaic circuit the work done in passing just through the plane of the circuit must be zero, we conclude that the potential of this plane must be $\pm 2i\pi$, and that in measuring the difference of potential for all other places we must remember that it will be $4\pi i$, greater or less according as the path pursued passes through the circuit or round outside it.

Prop. IV. The work done on a unit pole in bringing it from a point A at which the potential of a voltaic circuit is $i\Omega_1$ to a point B at which the potential is $i\Omega_2$ is

$$-i(\Omega_2 - \Omega_1),$$

assuming the path pursued not to go through the circuit. If the path pass through the circuit it is represented by

$$i(\pm 4\pi - \Omega_2 + \Omega_1),$$

MAGNETISM.

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the − or + being taken according as the path through the circuit is positive or negative relatively to the lines of force.

239. Prop. V A. The work done on a shell placed in a magnetic field, and moved from a position in which N_1 lines of force intersect it to a position in which N_2 lines of force intersect it, is

$$j(N_2 - N_1).$$

Art. 224.

Prop. V. The work on a voltaic circuit placed in the magnetic field, and moved from a place in which N_1 lines of force intersect it to a position in which N_2 lines of force intersect it, is

$$-i(N_2 - N_1).$$

240. Prop. VI A. If $N_2 < N_1$ the work done is negative, or the shell acquires kinetic energy owing to the magnetic forces helping the movement. Art. 224.

Prop. VI. If $N_2 > N_1$ the work done is negative, that is the circuit acquires kinetic energy owing to the electromagnetic forces assisting the movement.

241. Prop. VII A. A magnetic shell capable of movement in the magnetic field places itself so as to include the smallest possible number of lines of force, or, what is the same thing, the greatest possible number of negative lines of force. Art. 225.

Prop. VII. A voltaic circuit free to move places itself in the field so as to include the greatest possible number of lines of force. That is, it will place itself in the strongest part of the field in such a position that the lines of force are as nearly as may be perpendicular to it, the current being related to the direction of the lines of force in right-handed cyclical order.

242. Since the potential at a point depends on the solid angle subtended by the circuit, we see that the surfaces over which the potential is constant will emanate from the circuit and will form bowl-shaped surfaces having the circuit for their edge. A system of equipotential surfaces would be a system of such unclosed surfaces intersecting each other at finite angles in the given circuit.

243. In assigning numerical values to the surfaces, we must remember that the potential represents the work done in carrying a unit pole from the surface to an infinite distance, and this will depend on whether the path pursued passes through the circuit or round its edge, and if it passes through the circuit, on how many times it passes through the circuit always in the same direction. Hence we cannot assign a fixed value to a given equipotential surface, but a series of values differing by $4\pi i$. The work done on a unit pole in bringing it from a surface whose potential is V_1 to another whose potential is V_2 is

$$\pm 4n\pi i + V_1 - V_2,$$

where n is the number of times the path pursued passes through the circuit in the same direction; if that direction be with the lines of force we prefix the $-$ sign, and if against them the $+$ sign.

The path of the pole which in the last paragraph passed through the circuit n times without returning, may be said conveniently to be *linked* n times with the circuit.

244. The lines of force cut all equipotential surfaces at right angles, and will therefore be a system of oval curves with the conducting wire passing through them. In conformity with the convention just made we may say that the lines of force are linked with the circuit, and the circuit with any one of its lines of force may be conceived as two successive links in a common chain.

245. We have already stated that the current is related to the lines of force in right-handed cyclical order. If we now conceive the line of force as a closed curve and the circuit as a direction cutting through it, the positive direction of the line of force will be related to that of the current in right-handed cyclical order. Hence the property is mutual.

This rule is clearly equivalent to that usually given, that a figure swimming in the current which enters by its heels and leaves by its head, and looking towards the magnet, sees the north pole driven to its left.

If the pole be fixed, and the current free to move, it

is clear that the current will be driven round a north pole so that the figure in the current looking towards the pole is always moved towards its right hand. A convenient rule for remembering this direction, often useful in practice, is that *a figure swimming in the current, looking along the lines of force, will, with the conductor, be carried towards its left.*

246. The two following propositions clearly follow from the foregoing.

Prop. VIII. In computing the potential of any closed circuit we may substitute for it any closed circuit which is obtained by projecting the given circuit by means of lines of force.

For since lines of force never intersect except at a magnetic pole we cannot by this means alter the number of lines of force enclosed.

Cor. 1. In any movement of a conductor the change in potential produced by the movement of any part of the closed circuit parallel to lines of force, or parallel to planes containing the lines of force in that part of the field, is nil.

Cor. 2. For any sinuous conductor a straight one may be substituted.

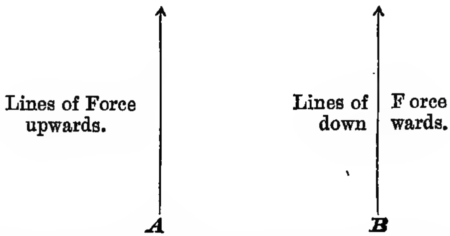
It is clear that a straight line can always be drawn through any sinuous line such that the number of lines of force omitted may be just counterbalanced by the number of extra lines of force included on substituting the straight for the sinuous current.

247. **Prop. IX. If two circuits more or less parallel to each other carry currents in the same direction they attract each other, and if the currents be in opposite directions they repel each other.**

Let A be a portion of a conductor carrying a current, and let the plane of the rest of the circuit be more or less perpendicular to the paper. Then it is clear (Art. 245) that the lines of force are a system of oval curves, rising from the paper to the left of A, and sinking into it to the right of A. If B be a parallel conductor carrying a current in the same direction, the lines of force enclosed by B, and on which the potential of B depends, will be all those which

fall to the right of B, and remembering the rule we see that B's current is positive to these lines of force.

Fig. 71.



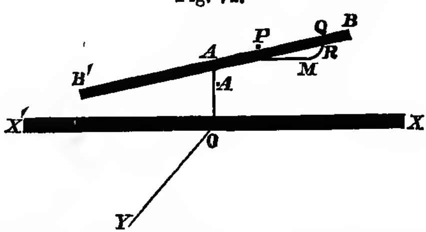
The electromagnetic action in the field will therefore (Art. 241) tend to place B so as to enclose more lines of force, that is, will draw B towards A. If the current in B be opposite to that in A, B's current will be negative to the lines of force, and the electromagnetic force will be therefore repulsive.

Cor. 1. If there be two straight wires parallel to each other carrying currents they will, if the currents be in the same direction, attract, and if in opposite directions, repel each other.

Cor. 2. If the two wires in the last corollary be inclined to each other and the currents both run into or out of the corner made by the wires, they will attract each other, but if one run into, and the other out of the corner, they will repel each other.

Let XOX' and BAB' be the two conductors, OA being

Fig. 72.



perpendicular to both. Let OY be at right angles to OX in a plane parallel to BAB'.

The lines of force due to XOX' will clearly be (neglecting its ends) a system of circles in planes perpendicular to it.

Let PQ be an element of BAB', and let it be projected into the bent line $PMRQ$ in which PM is parallel to OX, MR to OY, and RQ to OA. The parts QR, RM will be parallel to the plane containing the lines of force. Hence moving the conductor parallel to itself, we have only to consider the change in potential due to the movement of PM. If the currents in XOX' and BAB' both run into or out of the corner, PM and OX will be parallel currents in the same direction, and will attract each other. If one run into and the other out of the corner, the currents in PM and OX will be opposite, and they will therefore repel each other.

248. **Prop. X. If we have in a magnetic field two voltaic circuits A and B the number of A's lines of force which B encloses will be equal to the number of B's lines of force which A encloses, when the current-strength in each is unity.**

Let N_1 be the number of lines of force due to A enclosed by B, and N_2 the number due to B enclosed by A, and let i_1 and i_2 be the current-strengths in A, B respectively.

The work done in carrying B out of A's field of force will then be $N_1 i_2$.

The work done in carrying A so as to hold the same position in space relative to B will be $-N_2 i_1$.

Hence we see that an amount of work $N_1 i_2 - N_2 i_1$ would be expended in carrying a magnetic system against no external magnetic forces from one place to another, and this must vanish. Hence

$$N_1 i_2 - N_2 i_1 = 0,$$

or $$N_1 i_2 = N_2 i_1.$$

But since potential and consequently force at any point in the field of a voltaic circuit is, by Prop. 2, proportional to current-strength, we may make $N_1 = M_1 i_1$, and $N_2 = M_2 i_2$; we see then that

$$M_1 i_1 i_2 = M_2 i_1 i_2,$$

or $$M_1 = M_2.$$

But M_1 and M_2 will be the number of lines of force enclosed respectively by A and B when there is unit current in each. Hence the proposition.

249. Def. Coefficient of Mutual Induction. *The quantity* M *in the preceding proposition which gives the number of lines of force due to one of two circuits (each carrying unit current) enclosed by the other is defined to be the coefficient of mutual induction between them.*

If M be the coefficient of mutual induction between two circuits carrying currents i_1 and i_2 the potential of each due to the other's magnetic field will be Mi_1i_2. If the two circuits be free to move in the field they will clearly place themselves in such a position that M shall be as large as possible; this will be when the two circuits are as nearly as possible in the same plane and carry parallel currents.

250. Prop. XI. To find an expression for the whole energy in a circuit carrying a current.

That a voltaic current is a source of energy we have already seen, and when the circuit is separated from all other circuits its energy must clearly be kinetic. Whether this energy be that of moving electricity or of the movement of the conductor carrying the electricity, or of both combined, we cannot here enquire. In either case the analogy of the vis viva or kinetic energy of moving material bodies would lead us to conclude that it depends on the square of the current-strength. We see also that the potential energy of two circuits carrying currents depends on the geometry of the circuits and on the product of their current-strengths, and since all forms of energy must be of the same order, we may infer that the energy of a given circuit will be represented by a certain coefficient depending on the geometry of the circuit multiplied by the square of the current-strength.

In a given case let us call the coefficient L, and the current-strength i, the energy of the circuit will then be $L \cdot i^2$.

Place another conductor in all respects similar to the former, so as to coincide with it, and let it carry the same current in the same direction.

The energy of this system will clearly be represented by $L(2i)^2 = 4Li^2$, since the geometry is unaltered and the current doubled.

If the conductors be separated so as to be carried out of each other's field, their whole energy is reduced to the sum of their separate energies, or $2Li^2$.

Hence the work done in separating them is the difference, or $2Li^2$.

But by the previous Proposition, the work done in separating them is equal to the number of lines of force due to one enclosed by the other. When the circuits coincide each one encloses *all* the lines of force due to the other.

Let the quantity denoted by M, when the circuits coincide, be represented by M_0. This symbol then represents the whole number of lines of force embraced by the circuit carrying unit current. Hence the work done in separation is $M_0 i^2$;

$$\therefore \quad 2Li^2 = M_0 i^2,$$

$$L = \frac{1}{2} M_0.$$

DEF. COEFFICIENT OF SELF-INDUCTION. L *is defined to be the coefficient of self-induction of a circuit, and is equal to half the number of lines of force embraced by the circuit, when removed from all other circuits and carrying unit current.*

COR. If there be two circuits carrying currents, and if L, N be their coefficients of self-induction, and M the coefficient of mutual induction, the whole energy of the field when the current-strengths are i_1, i_2 is given by the expression

$$Li_1^2 \pm Mi_1 i_2 + Ni_2^2,$$

the positive or negative signs being given to the middle term as the lines of force from one circuit pass in the positive or negative direction through the other.

251. Having obtained the foregoing expression for the energy of a voltaic circuit carrying a current, we get the idea of inertia to be overcome in establishing the current at first, or making any alteration in it when once established.

On applying an electromotive force to a circuit, part of the energy of the battery is used up in overcoming the resistance or in heating the circuit, while the remainder goes to increase its kinetic energy.

252. It is perhaps a little beyond the scope of an elementary work like the present to give the equation for the establishment of the current, but as it follows from the preceding Article it will make the subject more complete if we supply it.

Using the notation we are now familiar with,

The energy derived from the battery in time $\tau = Ei\tau$, supposing τ a very short interval.

The energy given out in heat during the same time $= Ri^2\tau$.

The increase in kinetic energy $= L(i'^2 - i^2)$, if i' be the current-strength at the end, and i at the beginning, of the interval τ.

Hence by preceding Article,

$$\begin{aligned} Ei\tau &= Ri^2\tau + L(i'^2 - i^2) \\ &= Ri^2\tau + 2L\,i\,(i' - i)\,; \\ \therefore\ E\tau &= Ri\tau + 2L\,(i' - i) \ldots\ldots\ldots\ldots\ldots (1). \end{aligned}$$

It is easy to show* from this equation that for the current after a time t we have

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{2L}\cdot t}\right).$$

* This may be proved in the following elementary way:

Put (1) in the form

$$\left(\frac{E}{R} - i\right)\tau = \frac{2L}{R}(i' - i).$$

Let $\frac{E}{R} - i = y$, and $\frac{E}{R} - i' = y'$;

$$\therefore\ i' - i = -(y' - y);$$

$$\therefore\ y\tau = -\frac{2L}{R}(y' - y);$$

$$\begin{aligned} \therefore\ \tau &= -\frac{2L}{R}\frac{y' - y}{y} = -\frac{2L}{R}\log\left(1 + \frac{y' - y}{y}\right) \\ &= -\frac{2L}{R}(\log y' - \log y), \end{aligned}$$

The ratio $\frac{R}{2L}$ is generally so large that the rise in strength of the current takes place with such rapidity that we cannot observe it. In marine telegraphy, where the coefficient of self-induction is large and complicated by the Leyden-jar action of the insulating sheath, the term $\epsilon^{-\frac{R}{2L}\cdot t}$ leads practically to a lengthening out of the signal, so that a sharp signal transmitted to the wire by closing the circuit for an instant shows in a galvanometer at the other end a gradual rising and falling again of the current.

253. We may compare the establishment of a steady current in a conductor to the establishment of steady motion in a steam-engine. When the locomotive is moving along steadily the whole work done by the steam on the piston is used up in friction on the rails and in the machine. When the engine is quickening its pace the work done by the steam is greater than that used up in friction, and the difference goes to increase the kinetic energy of the system, and vice versa when the engine is pulling up.

So with electricity in motion. When steady Ohm's law expresses the fact that the energy given out by the battery is converted into heat in the circuit. When however the current is increasing the energy given out by the battery is more than that used up in the circuit, and the remainder goes to increase its kinetic energy, and vice versa when the current is ceasing. In practice the current becomes steady so rapidly that we only observe the indirect effect of the increasing energy in the extra spark.

254. We have seen that every voltaic circuit possesses an electromagnetic field, and in this field exerts attractions and

which is a form suitable for direct summation. Remembering that $y=\frac{E}{R}$ when $i=0$, we have after time t,

$$t=-\frac{2L}{R}\left\{\log\left(\frac{E}{R}-i\right)-\log\frac{E}{R}\right\};$$

$$\text{or}\quad \frac{\frac{E}{R}-i}{\frac{E}{R}}=\epsilon^{-\frac{R}{2L}\cdot t};$$

$$\therefore\ i=\frac{E}{R}\left(1-\epsilon^{-\frac{R}{2L}\cdot t}\right).$$

repulsions upon magnetic poles or other voltaic circuits. If in obedience to these attractions and repulsions movements take place, the law of conservation of energy shows us that the work done by the circuit must be done in some way at the expense of the energy in the circuit. This energy we have just seen to be kinetic, depending on the geometry of the circuit and the current-strength.

The only way therefore in which energy can be abstracted from or added to the circuit is by a diminution or increase of current-strength. The diminution of current-strength will last just long enough to compensate for the work done, and the steady current will be established again. These variations of current-strength in the circuit while work is being done in the electromagnetic field are known as induced currents. Since the induced current is always a compensation for the energy expended or gained in the field it is clear that acting alone it would oppose the movement, or its direction will always be such that by its electromagnetic effect it would oppose the movement taking place in the field. This is known as Lenz's Law.

From Lenz's law combined with the law stated in Art. 245, we have the rule for the direction of an induced current in a moving conductor, that *a figure in the conductor looking along the lines of force and moved towards his left with the conductor will experience an induced current which enters by his head and leaves by his heels.*

255. Prop. XII. To calculate the induced current produced by the movement of any conductor in a magnetic field.

During a very short time τ of the movement, the energy given out by the battery is $Ei\tau$, using the ordinary notation.

The amount expended in heat in the circuit is $Ri^2\tau$. If the number of lines of force cut through at the beginning of the movement be N, and at the end N', the energy acquired by the movement is $(N'-N)\,i$.

Hence

$$Ei\tau = Ri^2\tau + (N'-N)\,i,$$

$$\text{or}\quad E\tau = Ri\tau + (N'-N).$$

Let i_0 denote the steady current, then $E = Ri_0$. Hence

$$R(i - i_0)\tau = -(N' - N).$$

But $i - i_0$ is the induced current-strength and $(i - i_0)\tau$ is the quantity of electricity transmitted during the time τ. Adding for all the short intervals of the movement,

$$\Sigma(i - i_0)\tau = -\frac{\Sigma(N' - N)}{R}.$$

But $\Sigma(i - i_0)\tau$ is the quantity of electricity transmitted during the whole movement and may be denoted by $[i]$. This is often called the total induced current.

$\Sigma(N' - N)$ will be simply $N_1 - N_0$ when N_1 is the number of lines of force enclosed at the end, and N_0 the number at the beginning of the movement;

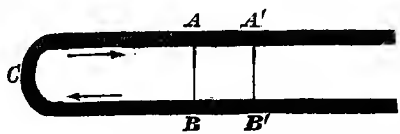
$$\therefore \quad [i] = -\frac{N_1 - N_0}{R}$$

$$= -\frac{\text{Number of lines of force added}}{\text{Resistance of circuit}}.$$

Cor. 1. If the increase of lines of force takes place at a uniform rate we shall have a uniform current whose intensity will be measured by $\frac{[i]}{t}$, where t is the time that the whole movement lasts.

Cor. 2. If a straight conductor forming part of a closed circuit is carried across lines of magnetic force, the electromotive force of the induced current is $-H.l.v$, where H is the number of lines of force per unit area or the strength of the field perpendicular to the plane of the conductor's motion, l the length of the conductor, and v the velocity with which it moves parallel to itself.

Fig. 73.



Let AB be the conductor, and let the rest of the circuit

be completed by thick bars A, C, B whose resistance may be neglected.

If the conductor move from AB to $A'B'$, and the lines of force be perpendicular to the paper, the number of lines of force added by the movement

$$= H \times \text{area } ABA'B' = H \times AA' \times l.$$

If $[i]$ denote the total current,

$$[i] = -\frac{H \times AA' \times l}{R}.$$

If the time occupied by the movement is t, the current

$$i = \frac{[i]}{t} = -\frac{H \times AA' \times l}{t},$$

the direction of this current being as in figure, if the lines of force are upwards.

But if the conductor move uniformly from AB to $A'B'$ in time t, $\frac{AA'}{t} = v$, the velocity of the motion;

$$\therefore\ i = -\frac{Hvl}{R};$$

$$\therefore\ Ri = -Hvl,$$

$$\text{and} \qquad Ri = E; \qquad \therefore\ E = -Hvl.$$

Cor. 3. From the preceding Cor., we see that the current will be strongest when the conductor is moved parallel to itself and perpendicular to the lines of force, the direction of the induced current being perpendicular to these two directions.

Cor. 4. We see also that there will be an induced current during the opening or closing of the circuit.

1st. *At closing the circuit.* Since L, the coefficient of induction, is half the number of lines of force enclosed by a circuit carrying unit current, the whole number added will be $2Li_0$, where i_0 is the steady current;

$$\therefore\ [i] = -\frac{2L}{R}\,i_0.$$

2nd. *At opening the circuit.* The number of lines *subtracted* will be $2Li_0$, hence

$$[i] = +\frac{2L}{R}i_0.$$

Since $$i_0 = \frac{E}{R}, \qquad [i] = \frac{2EL}{R^2},$$

and for the average electromotive force in either case

$$E' = R \times \frac{[i]}{t} = \frac{2EL}{Rt},$$

where t is the time the current lasts.

This result is interesting, since we see that by reason of the extreme smallness of t, E' may be many multiples of E. Thus the induced current having very high electromotive force is able to break across a finite air-space, giving rise to the galvanic spark or extra current, while no cell can break directly across an appreciable air-space. Sir W. Thomson says, that 5510 Daniell's cells would be required to give a spark between two brass terminals about $\frac{1}{8}$ in. apart.

256. Extending the analogy pointed out in Art. 253 we may compare induced currents with the phenomena attending the establishment of steady motion in a locomotive and train. As we all know at the first start the engine has to overcome the inertia of the train behind it, and as a consequence receives a number of impulsive jerks backwards just analogous to the induced negative current at closing the circuit. When the engine slackens pace it has to overcome the energy of the moving mass behind, and accordingly receives jerks forwards analogous to the induced positive current at opening the circuit.

257. We endeavoured in our third chapter to trace all electrical phenomena to a medium surrounding electrified bodies in a state of stress rather than to direct action at a distance. The phenomena of induced currents and indeed of attraction and repulsion between currents seem inexplicable on the theory of action at a distance, and in consequence Prof. Maxwell has shown by mathematical analysis

that all the phenomena of the magnetic field may be deduced by assuming a state of stress related to the lines of magnetic force, consisting of a tension along the lines of force in virtue of which they tend to shorten themselves, of pressure perpendicular to the lines of force causing them, if free to move, to repel each other, and of a rotational force about the direction of magnetization in all magnetized bodies. It is in order to familiarize the student as far as possible with these ideas that we have in this chapter preferred the electromagnetic to the better known electrodynamic method of Ampère, which does not lend itself so easily to the idea of action from molecule to molecule or to the fundamental principles of energy, though as Prof. Maxwell has shown it is not necessarily at variance with either of them.

Examples on Chapter VIII.

1. Two magnetic compasses are placed on a table near each other; explain how they influence each other's directions in all different positions.

2. A common bar magnet is placed on a table and a compass needle lies on the table subject to the magnet's force; show by a diagram the positions of the needle in different positions relatively to the magnet.

3. A dipping-needle is free to move in a plane at a given inclination to the magnetic meridian; show how to find the apparent dip.

4. Show that if the apparent dip observed in any two planes at right angles to each other be δ_1, δ_2, then δ the true dip can be found from the formula

$$\cot^2\delta = \cot^2\delta_1 + \cot^2\delta_2.$$

5. If the dipping-needle move in a plane perpendicular to the meridian, show that it will remain vertical. Hence show how to determine the plane of the meridian by observations with the dipping-needle only.

6. Show that in the case of a single bar magnet the lines of force are a system of oval curves which pass through both poles, and the equipotential surfaces are a system of unclosed surfaces which cut the axis between the poles.

7. If r, r' be the distances of a point from the north and south pole of a magnet respectively, show that for any point on a given equipotential surface $\frac{1}{r} - \frac{1}{r'}$ is of constant value.

8. If in the preceding question ψ, ψ' be the angles which r, r' make with the tangent plane to an equipotential surface, $\frac{\cos\psi}{\cos\psi'} = \frac{r^2}{r'^2}$.

9. If ϕ, ϕ' be the angles which r, r' in the preceding question make with a line of force, $\frac{\sin\phi}{\sin\phi'} = \frac{r^2}{r'^2}$.

10. If θ, θ' be the angles which r, r' make with the magnetic axis produced in one direction, show that along any line of force, $\cos\theta - \cos\theta'$ is constant.

11. Let two equal rods Nn, Ss turn on pivots about points N, S which are the poles of a given magnet. Then if they be moved so that ns is always perpendicular to NS, or NS produced, the intersection of Nn, Ss will trace out a magnetic curve. (See Roget's *Electricity*.)

12. Prove the following construction for obtaining any number of points on a system of magnetic curves:—Divide the magnetic axis into any integral number of parts, and set off along the axis produced any large number of equal parts. With centres N, S describe two equal circles having for any radii as large multiples of this subdivision as practicable, and divide their circumferences by perpendiculars drawn to the axis at each subdivision: draw lines joining N and S to the points of division of these circumferences; the lines of force will then be curvilinear diagonals of the lozenge-shaped spaces into which the figure is divided. (Roget's *Electricity*.)

13. Show that the equipotential surfaces will be closed

and the lines of force constantly divergent from either pole when the magnetic system consists of two similar and equal poles at a distance from each other.

14. Show that the lines of force for two similar poles will be the other set of curvilinear diagonals of the lozenge-shaped spaces indicated in ques. 12.

15. Find the form of the surface of zero potential for any bar magnet, and show that the resultant magnetic force for points on it is given by $\frac{ml}{r^3}$, where m is the strength of each pole, l the length, and r the distance of the point from one pole.

16. A closed voltaic circuit is supported at its centre of gravity but otherwise free. Explain the position assumed by it under the action of the earth's magnetism.

17. A straight conductor (capable of sliding freely on fixed bars and forming with them a closed voltaic circuit) carries a current. Explain the direction of its movement,

(i) When capable of moving parallel to itself in the horizontal plane and carrying a current from North to South.

(ii) When capable of moving parallel to itself in the horizontal plane and carrying a current from East to West.

(iii) When capable of moving parallel to itself in the vertical plane and carrying a current from North to South.

(iv) When capable of moving parallel to itself in the vertical plane and carrying a current from East to West.

18. A straight conductor carrying a current is capable of rotation round a magnetic pole. Show in all cases the relation between the sign of the pole, the direction of the current, and the direction of rotation.

19. Discuss the previous question in the case when the conductor is at rest and the magnetic pole free to move.

20. Show that a straight horizontal conductor placed East to West and carrying a current will, if exactly balanced, appear to lose or gain weight when the direction of the current is reversed. At what part of the earth will this effect be strongest?

21. A straight conductor carrying a current is fixed at one end, and the other rests by help of a cork-float in contact with mercury. Show that the conductor will rotate, but that except at the magnetic poles the rate of rotation in different parts of its course will not be the same.

22. A long wire carrying a current has a short straight wire also carrying a current perpendicular to it but not crossing it. Investigate the direction of movement (if any) in the short conductor for different directions of the currents.

23. Investigate the direction of rotation in Barlow's wheel for given directions of the lines of magnetic force and of the current.

24. A bar of soft iron has a coil of wire round it. Show by a diagram the direction of the current induced in the coil, (i) when a N. magnetic pole approaches one end of the bar, (ii) when the same pole is removed.

25. A bar magnet is drawn through a hollow coil of wire forming a closed circuit and back again. Show the direction, and roughly the variations in strength, of the induced current during the movement.

26. A straight wire forming part of a closed conductor is placed horizontally and slides parallel to itself, (i) from E. to W., (ii) from N. to S. Show in each case the direction of the current induced in it.

27. The same wire is arranged so that it can move in a vertical plane. Find the direction of the induced current, (i) when it rises vertically in a plane perpendicular to the meridian, (ii) when in the plane of the meridian.

28. Barlow's wheel (see ques. 23) has the battery removed but the battery circuit closed and the wheel made to rotate by mechanical means. Find the direction of the induced current.

29. A wire in the form of a closed circle rotates about a vertical axis in its own plane in the direction of the hands of a watch; investigate the direction of the induced current for different positions. Show that a small magnet suspended at the centre of the rotating coil will have its N. pole deflected towards the East if the rotation of the coil be with the hands of a watch. (See *B. A. Reports on Electrical Standards.*)

30. A wire parallelogram carrying a current is suspended from Ampère's stand, and allowed to take up its position of rest under the action of the earth alone. Explain its position of rest.

31. Show that a wire bent into the form of the figure 8 and carrying a current, will be astatic in relation to the earth.

32. A magnet is suspended horizontally over a diameter of a rotating copper disk. Show that the magnet will on rotating the disk be deflected from the meridian in the direction of rotation of the disk.

33. Explain the effect of a copper box on the oscillations of a magnet needle suspended within it.

34. A copper strip is drawn between the poles of a powerful horse-shoe magnet, and its opposite edges are connected by springs with galvanometer terminals. Show the direction of the induced current.

35. Show that the induced currents in the copper strip will retard the movement across the field.

36. Explain the difficulty of drawing a metal sheet between the poles of a powerful electromagnet.

37. A wire parallelogram is suspended on an Ampère's stand, and a circular copper plate rotated below it. Show the direction of the surface currents in the plate, and shew that the circuit will be deflected, following the direction of rotation of the plate.

38. If the conductor in the last question consist of two

equal parallelograms with the current flowing as in a figure 8, the system will rotate following the rotation of the plate.

39. A metal band of a circular form is made to turn on a vertical axis through its centre and perpendicular to its plane. Two points above each other on the upper and lower edge are connected by springs with the terminals of a galvanometer.

(i) A wire carrying a current is placed vertically near the springs; show that there will be an induced current, and investigate its direction.

(ii) The wire is bent into a circle and placed so as to surround the upper edge of the band; investigate the direction of the induced current.

(iii) Will there be a current when the bent wire encircles the band near its middle?

(iv) Extend this to the case of a conducting sphere which turns about a diameter and has the bent wire placed so as to embrace an equatoreal plane. Show that there will be *superficial* currents from the poles to the equator when the current and rotation are in the same direction.

(v) Show also the direction of the superficial currents when the bent wire is in a meridian of the revolving sphere.

CHAPTER IX.

ABSOLUTE DIMENSIONS OF PHYSICAL UNITS.

258. We stated in our first paragraph that all physical units are referable to the fundamental ideas of *space, time,* and *mass,* the units of which are arbitrary. These units once fixed, each definition we employ of a new unit contains implicitly its reference back to the absolute system. It is our object in this chapter to trace the measures of the units we have employed, and represent them in terms of arbitrarily assumed fundamental units.

259. We must remember that if we make any change in our units the change produced in the measure varies inversely as the change in the unit. Thus changing the unit of length from a foot to a yard, the measures of all distances will be divided by three, and the same principle applies in all cases.

260. There are two classes of units we have concerned ourselves with, *mechanical* units and *electrical* units, many of the latter having been referred to under two systems of measurement, the *electrostatic* and *electromagnetic.* We shall therefore divide our investigation into these three divisions, referring each time to the definition and the algebraical expression of it always implied.

We call the units of *length, time,* and *mass* respectively L, T, M.

261. (1) *Mechanical Units.*

Velocity is defined (Art. 2) as space described per unit of

time, and may therefore be measured by $\frac{\text{space}}{\text{time}}$. Hence retaining our former symbols as far as possible,

$$v = \frac{L}{T} = LT^{-1} \text{ .... (1).}$$

The meaning of this expression is, that given any change in the fundamental units (space and time), the change in the derived unit (velocity) is at once found by substituting in this formula the ratio between the new and old absolute units.

Acceleration is defined (Art. 5) as velocity added per second, and may be measured by $\frac{\text{velocity}}{\text{time}}$.

$$f = \frac{v}{T} = LT^{-2} \text{ .... (2).}$$

Momentum is defined (Art. 20) as the product of mass and velocity, and is therefore measured by

$$Mv = MLT^{-1} \text{ .... (3).}$$

Force is defined (Art. 20) as rate of change of momentum per second, and is measured by $\frac{\text{momentum}}{\text{time}}$.

$$F = \frac{MLT^{-1}}{T} = MLT^{-2} \text{ .... (4).}$$

Energy may be defined either as work (Art. 27), that is (force) × (space), or as vis viva (Art. 24), that is $\frac{1}{2}$(mass) × (velocity)^2. Denoting it by the general symbol W, we have in either case

$$W = ML^2T^{-2} \text{ .... (5).}$$

This will give the dimensions of every form of energy.

262. (2) *Electrostatic and Magnetic Units.*

Quantity. The measure of quantity depends ultimately on the law that the force between two equal quantities (Q) at a distance (L) from each other is measured by $\frac{Q^2}{L^2}$ (Arts. 55 and 68).

Hence, as far as dimensions are concerned,

$$\frac{Q^2}{L^2} = F = MLT^{-2} \quad\text{.....................(6)};$$

therefore $$Q = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}.$$

Magnetic pole, or *Quantity of Magnetism.* The same formula (Art. 207) expresses the force between two magnetic poles. Hence also

$$m = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \quad\text{.....................(7)}.$$

Electrical and Magnetic Density are defined (Arts. 59 and 206) as quantity of electricity or magnetism respectively per unit area, and are measured by $\frac{Q}{L^2}$ and $\frac{m}{L^2}$ respectively. Hence, in both cases,

$$D = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \quad\text{.....................(8)}.$$

Potential is defined (Arts. 62 and 207) as work done on a unit of electricity or magnetism, and is therefore measured by $\frac{W}{Q}$ or $\frac{W}{m}$ respectively. In both cases,

$$\left.\begin{matrix} V \\ E \end{matrix}\right\} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \quad\text{.....................(9)},$$

since Electromotive Force is a form of Potential.

Electrical or Magnetic Force is defined (Art. 65) as force on a plus unit, and may be measured by $\frac{F}{Q}$, which gives

$$\left.\begin{matrix} F \\ H \end{matrix}\right\} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \quad\text{...................(10)},$$

since in Magnetism the corresponding quantity is denoted by the symbol H, and is defined (Art. 207) as the number of lines of force per unit area.

Number of Lines of Force. Force denotes (Art. 207) the number of lines of force per unit area. Hence number of lines of force is measured by HL^2, or

$$N = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \quad\text{.....................(11)}.$$

Capacity is defined (Art. 69) as quantity per unit potential, and may be measured by $\frac{Q}{V}$. Therefore

$$C = L \quad\text{.........................(12)}.$$

Specific Inductive Capacity is defined (Art. 76) by the ratio of two quantities of electricity, and is an abstract number. Hence in dimensions

$$K = 0 \quad\text{.........................(13)}.$$

Current-Strength is defined (Art. 151) as quantity per second, and is measured by $\frac{Q}{T}$; therefore

$$I = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \quad\text{.....................(14)}.$$

Resistance is defined (Art. 161) by Ohm's law by the equation $E = RI$. Hence

$$R = \frac{E}{I} = \frac{T}{L} = L^{-1}T \quad\text{.................(15)}.$$

Conductivity is (Art. 156) the reciprocal of resistance, and is therefore measured by

$$LT^{-1} \quad\text{.........................(16)}.$$

Specific Conductivity is defined (Art. 150) by the formula $I = cF\sigma$.

$$c = \frac{\text{current-strength}}{(\text{force}) \times (\text{area})} = \frac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}} = T^{-1} \quad\text{.....(17)}.$$

Specific Resistance is defined as the reciprocal of specific conductivity, and is measured by

$$\rho = T \quad\text{.........................(18)}.$$

263. (3) *Electromagnetic Units.*

We shall denote electromagnetic measure by the same symbols as those employed in electrostatic, placing a bar over the symbol to indicate that it refers to this measurement.

Current-Strength. This is defined as being identical with the strength of a magnetic shell (Art. 239), which again is

defined (Arts. 217 and 218) as the product of magnetic density into length. Hence

$$\overline{I} = DL = M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \quad\text{.................}(19).$$

Quantity is defined by the formula $\overline{I} = \frac{\overline{Q}}{T}$. Therefore

$$\overline{Q} = \overline{I}\,.\,T = M^{\frac{1}{2}} L^{\frac{1}{2}} \quad\text{..................}(20).$$

Potential is defined (Arts. 207 and 231) as the work done on unit quantity, and is therefore measured by $\frac{W}{\overline{Q}}$. Therefore

$$\left.\begin{matrix}\overline{V}\\ \overline{E}\end{matrix}\right\} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \quad\text{..................}(21).$$

Resistance is defined (Art. 230) by Ohm's law. Therefore

$$\overline{R} = \frac{\overline{E}}{\overline{I}} = LT^{-1} \quad\text{.................}(22).$$

Capacity is defined as before by $\frac{\overline{Q}}{\overline{V}}$. Therefore

$$\overline{C} = \frac{M^{\frac{1}{2}} L^{\frac{1}{2}}}{M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}} = L^{-1} T^{2} \quad\text{..............}(23).$$

Electrical Force is force on unit quantity, and is therefore measured by $\frac{F}{\overline{Q}}$, which gives

$$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} \quad\text{......................}(24).$$

Number of Lines of Force may (Art. 239) be defined by the formula $\overline{N} \times \overline{I} =$ work: hence

$$\overline{N} = \frac{W}{\overline{I}} = \frac{ML^{2}T^{-2}}{M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}} = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \quad\text{..........}(25).$$

Coefficient of Mutual or Self-Induction is defined (Arts. 249, 250) by the formula $Li^{2} =$ energy of circuit; hence

$$\overline{L} = \frac{W}{\overline{I}^{2}} = L \quad\text{....................}(26).$$

Specific Conductivity is defined as before by the relation

$$\bar{c} = \frac{\bar{I}}{\bar{F}L^2} = L^{-2}T \quad \text{.................} (27).$$

Specific Resistance is defined as the reciprocal of the last, and therefore

$$\bar{\rho} = L^2T^{-1} \quad \text{....................} (28).$$

Specific Inductive Capacity may be defined by Art. 178, in agreement with the formula $\overline{KC} = \frac{1}{4\pi}\bar{\rho}\bar{K}$, whence

$$\bar{K} = \frac{T}{\bar{\rho}} = L^{-2}T^2 \text{.................} (29).$$

264. We will now present in a tabular view the results of the preceding articles, adding the ratio between the dimensions of the various units where they have been measured in both the electrostatic and the electromagnetic systems, this ratio being the number of electrostatic in one electromagnetic unit.

(1) Mechanical.

Unit.	Symbol.	Dimensions.
Velocity	v	LT^{-1}
Acceleration	f	LT^{-2}
Force	F	MLT^{-2}
Energy...........	W	ML^2T^{-2}

265. (2) Electrostatic and electromagnetic units.

Units.	Symbol.	Dimensions in Electrostatic Measure.	Dimensions in Electromagnetic Measure.	Ratio of Electrostatic to Electromagnetic Measure.
Quantity	Q	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}$	LT^{-1}
Potential Electromotive Force	V E	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$L^{-1}T$
Electric Force	F	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}$	$L^{-1}T$
Capacity	C	L	$L^{-1}T^{2}$	$L^{2}T^{-2}$
Specific inductive capacity	K	0	$L^{-2}T^{2}$	$L^{2}T^{-2}$
Specific conductivity	c	T^{-1}	$L^{-2}T$	$L^{2}T^{-2}$
Specific Resistance	ρ	T	$L^{2}T^{-1}$	$L^{-2}T^{2}$
Resistance of a Conductor	R	$L^{-1}T$	LT^{-1}	$L^{-2}T^{2}$
Current strength	I	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	LT^{-1}
Magnet pole Quantity of Magnetism	m	——	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	——
Magnetic potential	V	——	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	——
Magnetic Force Strength of Field	H	——	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	——
Number of Lines of Force	N	——	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	——
Coefficient of Mutual or Self-induction	L		L	

266. In the above table we see that the ratio of the measures of each magnitude in the two systems depends on LT^{-1}, a quantity of the dimensions of a velocity. This velocity has been determined by experiment, and must be an absolute quantity independent of any particular system of measurement. We will denote it by v, which may be assumed equal to 300 million metres per second. From the preceding table we clearly have for the ratios between measures in the two systems

$$\frac{Q}{\overline{Q}}=\frac{I}{\overline{I}}=\frac{\overline{V}}{V}=\frac{\overline{F}}{F}=v,$$

$$\text{and}\quad \frac{C}{\overline{C}}=\frac{K}{\overline{K}}=\frac{c}{\overline{c}}=\frac{\overline{\rho}}{\rho}=\frac{\overline{R}}{R}=v^{2}.$$

267. If the suffix 0 denote that the symbols stand for units instead of measures, we shall have for the ratio between the units themselves by Art. 259

$$\frac{Q_0}{\overline{Q}_0}=\frac{I_0}{\overline{I}_0}=\frac{\overline{V}_0}{V_0}=\frac{\overline{F}_0}{F}=\frac{1}{v},$$

$$\text{and}\quad \frac{C_0}{\overline{C}_0}=\frac{K_0}{\overline{K}_0}=\frac{c_0}{\overline{c}_0}=\frac{\overline{\rho}_0}{\rho_0}=\frac{\overline{R}}{R_0}=\frac{1}{v^{2}}.$$

268. *Practical units* are not the absolute units given above, immediately derived from the C. G. S. system of measurement. It is found that by choosing these units, all our resistances and electromotive forces will be represented by very large numbers, and all our quantities and capacities by small fractions. The units of length, time, and mass actually taken are these:

For length, the quadrantal arc of the earth or 10^9 cm.

For time, the second remains the unit.

For mass, the 10^{-11} gm. is chosen.

Referring to the table of dimensions given above we see that electromotive force $(M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2})$ becomes multiplied by $10^{-\frac{11}{2}} . 10^{\frac{27}{2}} = 10^8$. The *volt* is therefore defined as 10^8 absolute electromagnetic units.

Resistance, whose dimensions are LT^{-1} (the same as those of a velocity), becomes multiplied by 10^9. The *ohm* is defined as the resistance which is represented by the velocity of 10^9 cm. per second.

Quantity, whose dimensions are $L^{\frac{1}{2}} M^{\frac{1}{2}}$, is multiplied by 10^{-1}. The *farad* is defined as representing 10^{-1} units of quantity.

Capacity, whose dimensions are $L^{-1} T^2$ is multiplied by 10^{-9}. The *farad*, which is also the term applied to the unit of capacity, will therefore represent 10^{-9} absolute C. G. S. units of the electromagnetic system.

Thus we see that to convert into absolute electromagnetic units of the C. G. S. system,

the volt; ohm; farad (quantity); farad (capacity);

we multiply by respectively

$$10^8, \; 10^9, \; 10^{-1}, \; 10^{-9}.$$

Examples on Chapter IX.

1. Make the following conversions by means of the formulæ given in this chapter.

(1) Find the number of ergs in a foot-pound.

Ans. 421383 nearly.

(2) Convert the acceleration of gravity (32·2) into the C. G. S. system. *Ans.* 981 nearly.

(3) The units of time, space, and mass, being changed from a second, foot, and pound to a minute, yard, and cubic foot of mercury (density 13); find the ratio of the new units of velocity, force and work, to the old units respectively.

Ans. $\frac{1}{20}, \frac{65}{96}, \frac{65}{32}$.

2. Show that in electromagnetic measure the dimensions of current strength are given by $\frac{\text{magnet pole}}{\text{length}}$. Hence show that the electromagnetic attraction between two conductors carrying currents will be of the same dimension as the product of the current strengths.

3. By comparing the attraction between two currents with that between two quantities of electricity condensed in points, show that the ratio between the electromagnetic and electrostatic units of quantity is represented by a velocity.

4. Show also that a current in electromagnetic measure is of the same dimensions as magnetic potential.

5. A sphere is raised to a given potential and discharged through a wire, the sphere contracting during the discharge so that its potential remains constant; show that the rate of contraction of the sphere will be equal to the reciprocal of the resistance of the wire in electrostatic measure.

6. Show that the heat given out in any circuit is expressed by the same formula whether the units be electrostatic or electromagnetic.

7. A coil whose resistance is 2 ohms is immersed in a kilogramme of water and a current of 3 farads passes through it for a minute. Find the elevation in temperature of the water (assuming the mechanical equivalent of heat to be 41560000 ergs per gm.-degree). *Ans.* $\frac{1}{4}^{\circ}$ C. nearly.

8. Find the radius of a sphere whose electrostatic capacity is one farad. *Ans.* 3×10^{9} cm.

9. The electrostatic capacity per nautical mile of any gutta-percha cable is found to be $\dfrac{\cdot 18769}{\log \dfrac{D}{d}}$ farads, and the resistance of its insulating sheath $\dfrac{\log \dfrac{D}{d}}{13} . 10^6$ ohms. Calculate the time of falling to half charge. (Given $\log_e 2 = \cdot 6931471$.)

Ans. 2 hrs. 46 min. nearly.

10. The resistance of gutta-percha is to that of Hooper's material as 1 to 16, and the specific inductive capacity as 4·2 to 3·1. Find from the last result the time of falling to half charge in a condenser of Hooper's material.

Ans. 32 hrs. 48 min.

11. Two plates whose areas are each one sq. cm. being placed at a distance of 2 mm. apart and connected with the terminals of a battery, are found to exert on each other a force equal to ·01gm. Find in electrostatic and electromagnetic measure the electromotive force of the battery.

Ans. 3·1 and 93×10^{10} nearly.

12. Show that the electromotive force in the preceding question is nearly equal to that of 930 Daniell's cells.

13. A metre is defined to be a ten-millionth part of the quadrantal arc of the earth; show that the electrostatic capacity of the earth is about $\dfrac{2}{3\pi}$ of a farad.

14. Express in absolute electromagnetic units the capacity of the earth.

Ans. $\dfrac{2}{9\pi \times 10^{11}}$.

CHAPTER X.

PROBLEMS IN MAGNETISM.

269. THE following five propositions on the properties of bodies free to move about a fixed axis, which might have been placed in Chap. I., will be found of service in this chapter, when treating of the motion of suspended magnets.

270. The idea we have here to introduce is that of *angular motion*, which can be uuderstood by fixing on a line in the body perpendicular to its axis and showing the angle traced out during the motion, the rate at which this angle is traced out being the angular velocity.

DEF. *The angular velocity of a rotating body is the angle traced out per second by a line fixed in the body perpendicular to the axis of rotation.*

The angular velocity like ordinary velocity is a property of the body at a particular instant, and if variable must be measured by the angle which would be traced out per second, supposing the angular velocity to remain constant.

271. Prop. I. If a body be rotating with angular velocity ω the velocity of a particle in the body distant r from the axis of rotation is $r\omega$.

For each particle traces out a circle and if θ be the angle traced out in a small time τ, the angular velocity will be $\frac{\theta}{\tau}$; or $\theta = \omega\tau$. The length of the arc traced out by the particle is $r\theta$, and the velocity of the particle $\frac{r\theta}{\tau}$ or $r\omega$.

272. Prop. II. To find the energy of a body rotating with angular velocity ω.

The energy of the particle whose velocity we computed in the last article is $\frac{1}{2}mv^2$ when m is its mass, and this is equal to $\frac{1}{2}mr^2\omega^2$. Hence the energy of the whole body is

$$\frac{\omega^2}{2} \cdot \Sigma mr^2.$$

Σmr^2 depends only on the density and shape of the body, and may be computed when the form of the body is known. It is defined to be the *moment of inertia*, and may be denoted by the symbol M. The energy of the rotating body we can then write $\frac{1}{2}M\omega^2$.

273. Prop. III. To find the angular velocity imparted to a body by a couple acting for a given time.

Let F be the force and l the arm of the couple. The work done by the force in twisting the body through a small angle θ is $F \times l\theta$.

If ω_1 and ω_2 be the angular velocities at the beginning and end of the movement the energy imparted is

$$\tfrac{1}{2}M(\omega_2^2 - \omega_1^2);$$
$$\therefore \tfrac{1}{2}M(\omega_2^2 - \omega_1^2) = F \times l\theta = G\omega_1\tau,$$

supposing G the moment of the couple and τ the time occupied by the movement,

$$\therefore M(\omega_2 - \omega_1) = G\tau.$$

If the couple remain constant for any finite time t and ω be the whole angular velocity imparted

$$M\omega = Gt.$$

274. Prop. IV. To find the elongation of swing of a body acted on by a constant force in a given direction (e.g. a pendulum under gravity).

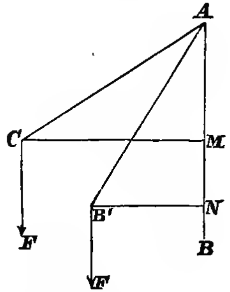
Let AB be the plumbline in the position of equilibrium, and AC the limit of its swing, then CAB is the elongation required.

If B' be an intermediate position the work spent between B and B' will be $F.BN = Fl(1-\cos\theta) = G(1-\cos\theta)$, where $\theta = BAB'$ and $Fl = G$.

Hence if ω_0 be the angular velocity at B and ω at B',

$$\tfrac{1}{2} M (\omega_0^2 - \omega^2) = G (1 - \cos\theta).$$

Fig. 74.



If $CAB = \alpha$, then when $\theta = \alpha$, $\omega = 0$;

$$\therefore \tfrac{1}{2} M\omega_0^2 = G (1 - \cos\alpha) = 2G \sin^2\frac{\alpha}{2};$$

$$\therefore \omega_0 = 2 \sin\frac{\alpha}{2}\sqrt{\frac{G}{M}},$$

which gives the relation required.

275. **Prop. V. To find the time of oscillation about the position of rest of the body in the preceding article, the disturbing force being supposed small.**

Let, as before, AB be the position of equilibrium, AC and AC' the extreme elongations.

For the angular velocity at any intermediate position AP, we shall have by the preceding Article

$$\tfrac{1}{2} M\omega^2 = G (\cos\theta - \cos\alpha),$$

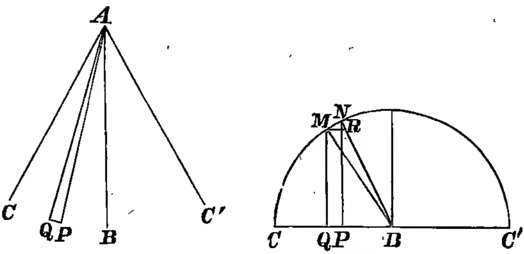
where $\theta = PAB$ and $\alpha = CAB$.

If the disturbing force be very small the elongation will also be small, and we may put $\cos\theta = 1 - \dfrac{\theta^2}{2}$ and $\cos\alpha = 1 - \dfrac{\alpha^2}{2}$;

$$\therefore M\omega^2 = G (\alpha^2 - \theta^2).$$

Set off the double elongation CBC' on a straight line so that $CC' = 2\alpha$, and $BP = \angle BAP$ and $BQ = \angle BAQ$. On CC'

Fig. 75.



describe a semicircle and raise perpendiculars, PN, QM, and draw MR parallel to BC. Join BM, BN. Then

$$M\omega^2 = G(\alpha^2 - \theta^2)$$
$$= G(BN^2 - BP^2) = G \,.\, PN^2;$$
$$\therefore \ \omega = \sqrt{\frac{G}{M}} \,.\, PN.$$

Hence time of describing the arc PQ

$$= \frac{P\hat{A}Q}{\omega} = \frac{PQ}{\sqrt{\frac{G}{M}} \,.\, PN} = \sqrt{\frac{M}{G}} \,.\, \frac{PQ}{PN}.$$

But by similar triangles $PQ : MN :: PN : BN$,

$$\therefore \ \frac{PQ}{PN} = \frac{MN}{BN} = \angle MBN;$$

$$\therefore \text{ time of describing } PQ = \sqrt{\frac{M}{G}} \,.\, M\hat{B}N.$$

Adding all the successive intervals we shall have the time of describing $BC = \frac{\pi}{2}\sqrt{\frac{M}{G}}$.

Hence if T be the time of a complete oscillation from rest to rest

$$T = \pi\sqrt{\frac{M}{G}};$$
$$\therefore \ GT^2 = M\pi^2.$$

COR. 1. We see from the result that the time of oscillation is independent of the arc of vibration, supposing this arc small. The vibrations are in consequence said to be isochronous.

COR. 2. If the pendulum consist of a mass m suspended from a weightless string of length l, under gravity $M = ml^2$ and $G = mlg$; hence $T = \pi\sqrt{\frac{l}{g}}$.

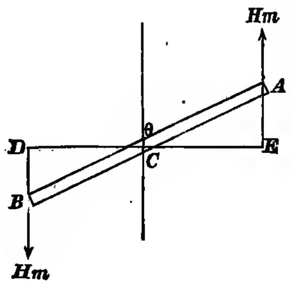
276. We now append some important applications of the preceding theory.

Prop. VI. To find the force acting on a magnet placed in a uniform magnetic field.

At a great distance from a magnetic system we may for a limited space consider the rate of change of potential or the magnetic force to be uniform. The lines of force will then be a system of parallel lines uniformly distributed through the space. This will apply for instance in the case of the earth throughout an ordinary room in which we perform our experiments. The direction of the lines of force are then shown by the dipping-needle and the thickness of their distribution by the absolute value of the magnetism at the place under consideration.

Let now the lines of force be parallel to the length of the paper, and let their direction be from the bottom to the top of the page.

Fig. 76.



Let AB be a magnet whose north pole is at A and south pole at B, and let the strength of each pole be m. Let l be the length of the magnet, and θ its inclination to the lines of force.

Let H be the intensity of the field, that is the force with which a unit pole would be urged along the lines of force. Then the pole A will be subject to a force Hm along the lines of force, and B will be subject to an equal and opposite force $-Hm$ along the lines of force.

These two forces constitute a couple (Art. 22), and the moment of the couple is $Hm \,.\, DE$.

But $$DE = l \sin \theta.$$

Hence the force on the magnet will be a couple whose moment is

$$Hml \sin \theta.$$

Cor. If the magnet be placed perpendicular to the lines of force the moment of the forces acting on it becomes Hml.

277. Def. *The product of the intensity of each pole of a magnet with the distance between the poles is called the 'magnetic moment' of the magnet.*

The moment of the forces acting on a magnet placed perpendicular to the lines of force in a field whose intensity is H will be HG, where G is the magnetic moment of the magnet and H the strength of the field.

278. Prop. VII. To calculate the swing of a magnet placed in a magnetic field and making oscillations about its position of rest.

The calculation of the preceding articles applies, and if G be the magnetic moment, M the moment of inertia, ω the angular velocity in the position of rest, α the greatest elongation, and H the strength of the field, then (Art. 274)

$$M\omega^2 = 2HG(1 - \cos \alpha),$$

$$\text{or} \quad \omega = 2 \sin \frac{\alpha}{2} \sqrt{\frac{GH}{M}}.$$

279. Prop. VIII. To calculate the time of oscillation of a magnet making small vibrations about its position of rest.

The calculation of Art. 275 applies if we substitute for G, HG.

Hence $$HGT^2 = \pi^2 M.$$

Cor. 1. Since G and M depend only on the magnet used, we may by observing the time of vibration calculate H.

Thus if the magnet make n oscillations per minute,

$$T = \frac{60}{n};$$

$$\therefore \quad H = \frac{\pi^2 M}{G} \cdot \frac{n^2}{60^2}.$$

Hence we see that if the magnet remain unaltered the strength of the field or the intensity of magnetic force at a place varies as the square of the number of vibrations made in any given time.

Cor. 2. In the above formula G depends on the strength of the magnetic pole and M only on the geometry of the magnet. The intensity of the magnetic pole owing to induction is constantly varying and cannot be taken as constant for a day. This formula, however, gives us the product HG in absolute measure.

280. By using one of the poles of the magnet to deflect another magnet we can find by observation the ratio $\frac{H}{G}$.

Let m be a pole of the magnet of the last article, and BC another magnet disturbed by it from MM the plane of the meridian.

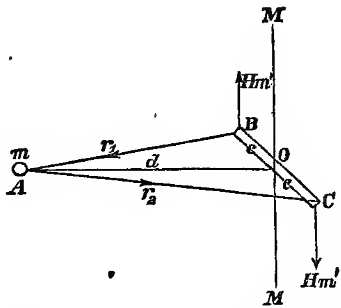
Let the intensity of the poles of BC be each m', its length $2c$, the distance of its middle point from A, d, and the distances AB, AC of the poles r_1 and r_2. The forces

acting on B will be Hm' along the meridian and $\frac{mm'}{r_1^2}$ along BA: the forces on C being $-Hm'$ along the meridian and $\frac{mm'}{r_2^2}$ along AC.

The force $\frac{mm'}{r_1^2}$ can be resolved into (Art. 17)

$$\frac{mm'}{r_1^2} \cdot \frac{d}{r_1} \text{ along } AO, \quad \text{and} \quad \frac{mm'}{r_1^2} \cdot \frac{c}{r_1} \text{ along } BO.$$

Fig. 77.



The force $\frac{mm'}{r_2^2}$ can be resolved into

$$\frac{mm'}{r_2^2} \cdot \frac{d}{r_2} \text{ along } AO, \quad \text{and} \quad \frac{mm'}{r_2^2} \cdot \frac{c}{r^2} \text{ along } CO.$$

Taking moments about O, if δ be the deflection from the meridian, we have

$$c \cos \delta \left(\frac{cmm'}{r_1^3} + \frac{cmm'}{r_2^3}\right) = 2c \sin \delta Hm';$$

$$\therefore \quad \frac{H}{m} = \frac{c \cot \delta}{2} \left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right).$$

Now $r_1^2 = c^2 + d^2 - 2cd \sin \delta$

$$= d^2 \left(1 - \frac{2c \sin \delta}{d} + \frac{c^2}{d^2}\right),$$

and
$$r_2^2 = d^2 \left(1 + \frac{2c \sin \delta}{d} + \frac{c^2}{d^2}\right).$$

Therefore $r_1^{-3} = d^{-3}\left(1 + \frac{3c \sin \delta}{d} + \dots\dots\dots\right)$,

and $r_2^{-3} = d^{-3}\left(1 - \frac{3c \sin \delta}{d} + \dots\dots\dots\right)$,

neglecting powers of $\frac{c}{d}$ above the first;

$$\therefore \quad \frac{1}{r_1^{\,3}} + \frac{1}{r_2^{\,3}} = \frac{2}{d^3};$$

$$\therefore \quad \frac{H}{m} = \frac{c \cot \delta}{d^3}, \qquad \text{or} \qquad \frac{H}{G} = \frac{c \cot \delta}{ld^3},$$

which gives in absolute measure the ratio $\frac{H}{G}$.

Knowing (Art. 279) the product HG the values of H and G are known in absolute measure. This method is due to Gauss, and is that now universally employed for determining the intensity of Terrestrial Magnetism in absolute units.

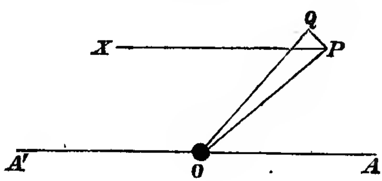
The quantity H actually measured is the horizontal component of the earth's magnetism, since a horizontally suspended magnet is more suitable than a dipping-needle for observing oscillations.

281. Prop. IX. To investigate the properties of the electromagnetic field near a straight wire of infinite length carrying a current.

We may regard the infinite wire as part of a circuit, the rest of the circuit lying in a plane which we will assume perpendicular to the plane of the paper, as also the conductor under consideration which forms one edge of the circuit.

Let O be a section of the conductor, OA' the plane of the circuit, and P the given point.

Fig. 78.



The circuit being of infinite extent above and below the paper and to the left of O, will now subtend a solid angle at P which has for its boundary a plane passing through the conductor (whose projection is PO) and a plane passing through the opposite part of the circuit. This part is parallel to the given conductor and at an infinite distance, and the plane passing through it will be parallel to the plane of the circuit and will have for its projection PX, which is parallel to OA'.

The solid angle bounded by the two planes PO, PX will be a lune of the unit sphere and its area will clearly be

$$\frac{\text{circular measure of } O\hat{P}X}{2\pi} \times \text{area of sphere}$$

$$= \frac{\theta}{2\pi} \times 4\pi = 2\theta,$$

where $$\theta = \angle POA.$$

Hence the potential at P is $2\theta i$.

This expression shows that for all points on the plane OP the potential is the same. Hence the equipotential surfaces are a system of planes intersecting in the conductor, and the lines of force are consequently a system of circles in planes parallel to the paper having O for their centre.

We must also remember that each of these surfaces has not a definite potential $2\theta i$ but its general potential will be

$$(4n\pi + 2\theta)\, i.$$

282. To find the strength of the field or the magnetic force at P we must find the rate of change of potential along a line of force. If PQ be an arc of a circle whose centre is O, it will be an arc of the line of force through P. If H be strength of field,

$$H = \frac{\text{Potential at } Q - \text{Potential at } P}{PQ}$$

$$= \frac{2P\hat{O}Q\, i}{PQ} = \frac{2\, i}{OP}; \text{ since } OP \,.\, P\hat{O}Q = PQ.$$

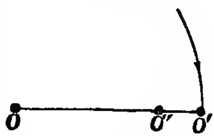
This gives us the strength of the field at every point round the conductor; the direction of the force being always

perpendicular to a plane containing the conductor and the magnet pole.

COR. 1. To find the attraction between the infinite conductor of this problem and another of finite length placed parallel to it and carrying a current.

Let O' be the trace on the paper of the second conductor of length l which will also be perpendicular to the paper, and let it carry a current i'.

Fig. 79.



If the conductor be moved parallel to itself to O'', the increase in the lines of force enclosed will be $H \times l \times O'O''$, when H is the strength of the field. Here $H = \dfrac{2i}{OO'}$.

the energy gained by the movement

$$= \frac{2il \, . \, O'O''}{OO'} i' = \frac{2lii'}{OO'} \, . \, O'O'';$$

Hence the rate of change of potential or attraction between the two conductors $= \dfrac{2lii'}{OO'}$.

COR. 2. The induced current in the second conductor owing to moving it parallel to itself, assuming that in the primary unaltered by the movement,

$$= -\frac{1}{R} \Sigma \frac{2liO'O''}{OO'} = + \frac{2li}{R} \Sigma \log \left(1 - \frac{O'O''}{OO'}\right)$$

$$= \frac{2li}{R} \Sigma \left(\log OO'' - \log OO'\right).$$

R being the resistance in the circuit.

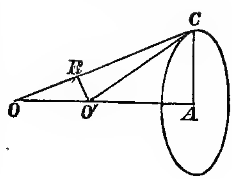
Hence during any finite movement, as from O_1 to O_2

$$[i] = -\frac{2li}{R} \log \frac{OO_2}{OO_1}.$$

283. Prop. X. To investigate the magnetic field along the axis of a circular voltaic circuit.

Let O be a point on the axis and CAC' the plane of the circuit.

Fig. 80.



To find the potential at O we have only to compute the solid angle subtended by the circuit at O. This will be the base of a right cone whose semi-vertical angle is COA.

It is easy to see that the area of the unit sphere cut off by this cone is $2\pi(1-\cos\theta)$ where $\theta = C\hat{O}A$.

The potential at O is consequently $2\pi i(1-\cos\theta)$.

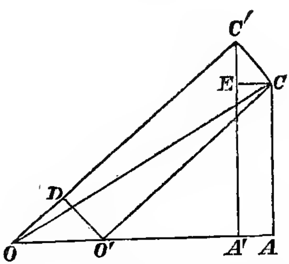
If the point be off the axis the cone becomes oblique, and there is no means of estimating its solid angle exactly.

284. To find the strength of the field at O we must compute the rate of change of potential in the direction OO' This will be

$$\frac{2\pi i(\cos\theta - \cos\theta')}{OO'},$$

where $CO'A = \theta'$.

Fig. 81.



To exhibit this geometrically, draw OC' parallel to $O'C$ and describe a circular arc CC'. Draw CA, $C'A'$ perpendicular to OA, CE perpendicular to $C'A'$, and $O'D$ perpendicular to OC'.

$$\text{Then } \cos\theta - \cos\theta' = \frac{OA}{OC} - \frac{OA'}{OC'} = \frac{OA - OA'}{OC} = \frac{AA'}{OC}$$
$$= \frac{CE}{OC} = \frac{CE}{CC'}\cdot\frac{O'D}{OO'}\cdot\frac{OO'}{OC},$$

remembering that CC' is approximately parallel to $O'D$.

But by similar triangles $\dfrac{CE}{CC'} = \dfrac{AC}{O'C}$ and $\dfrac{O'D}{OO'} = \dfrac{C'A'}{OC'}$.

Hence $\dfrac{2\pi i(\cos\theta - \cos\theta')}{OO'} = 2\pi i \dfrac{AC}{O'C} \cdot \dfrac{A'C'}{OC'} \cdot \dfrac{1}{OC}$, which when O, O' are very near becomes $\dfrac{2\pi i AC^2}{OC^3} = H$, the strength of the magnetic field at O.

Cor. 1. The potential at the centre is $2\pi i$, and the strength of field there is $\dfrac{2\pi i}{AC}$.

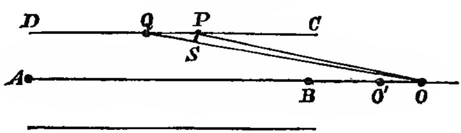
Cor. 2. If the wire be wound n times round the circle we consider separately the strength of field due to each coil and add them. The result will be that the potential and strength of field are each multiplied by n.

285. Prop. XI. To investigate the strength of field due to a helix carrying a voltaic circuit. Such an arrangement is often called a solenoid*.

We shall regard the helix as made up of a number of parallel circles to each of which the proposition of the preceding article may be applied.

Let AB be the axis of the helix, PQ an element of its surface, and O a point on the axis produced.

Fig. 82.



The strength of current in the element PQ will be $ni \,.\, PQ$, supposing n the number of layers of the wire per unit length.

* The term solenoid is only strictly applied to a helix in which the terminal wires are carried back again to the centre and leave the helix side by side. These return currents would (Art. 246, Cor. 2) neutralize the helix regarded as a sinuous current about its own axis, and we have left the action of a system of circles in parallel planes as assumed here.

Hence the potential of the annulus PQ on O

$$= 2\pi niPQ\,(1-\cos\theta),\text{ when } POA=\theta,$$
$$= 2\pi ni\,(PQ - PQ\cos\theta)$$
$$= 2\pi niPQ - 2\pi ni\,QS;$$

where PS is perpendicular to OQ,

$$= 2\pi niPQ - 2\pi ni\,(OQ - OP).$$

The potential of the whole helix is found by summing all these elementary potentials and may be written

$$2\pi ni\,\Sigma PQ - 2\pi ni\,\Sigma\,(OQ - OP),$$
$$\Sigma PQ = \text{the whole length of the helix} = AB,$$
$$\Sigma\,(OQ - OP) = OD - OC.$$

Hence the potential becomes

$$2\pi niAB - 2\pi ni\,(OD - OC).$$

286. To find the strength of the Electro-magnetic field at O we must find the rate of change of potential from O along OB. The rate of change in the first term $(2\pi ni\,AB)$ we can see vanishes since AB is a fixed length not depending on O.

The rate of change in the second term

$$= 2\pi ni\,\frac{OD - OC - O'D + O'C}{OO'}$$
$$= 2\pi ni\left(\frac{OD - O'D}{OO'} - \frac{OC - O'C}{OO'}\right).$$

But in fig. 80 we see that

$$\frac{OC - O'C}{OO'} = \cos COA = \cos\psi \text{ suppose},$$

and similarly $\dfrac{OD - O'D}{OO'} = \cos DOA = \cos\psi'$;

supposing $DOA = \psi'$.

Hence the strength of the field at O or H

$$= 2\pi ni(\cos\psi' - \cos\psi).$$

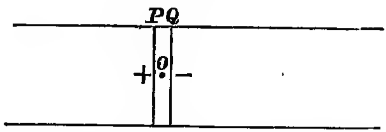
If O be inside the helix the angle COA becomes obtuse, and we shall have, if ψ, ψ' denote the acute angles made with the axis,

$$H = 2\pi ni(\cos\psi + \cos\psi').$$

If the diameter of the helix be very small compared to its length $\cos\psi$ and $\cos\psi'$ will for a large fraction of the length be both nearly unity, and the strength of the field within the helix becomes constant and equal to $4\pi ni$.

287. This last result might have been obtained by substituting for each circle its equivalent plane circular magnetic shell. The positive side of one shell will lie against the negative side of the next, and on external points their influence is nil. Hence the effect of the helix will on the whole be the same as that of the imaginary distribution of magnetism on its ends. For the force on an internal point we must consider in addition the force due to the magnetic distribution over the cavity in which the point lies (Art. 211). This cavity may be made by the removal of an elementary shell.

Fig. 83.



Let O be the point, and PQ the elementary shell. The potential in passing through O changes (Art. 237) from $2\pi niPQ$ at P to $-2\pi niPQ$ at Q. Hence the force

$$= \frac{2\pi niPQ - (-2\pi niPQ)}{PQ} = 4\pi ni.$$

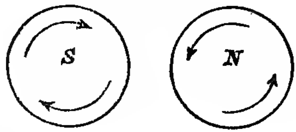
This shows that, neglecting the distribution over the ends, the force is the same everywhere within the helix and not only along its axis.

Cor. 1. A wire in the form of a helix traversed by

a voltaic circuit, may in all cases be substituted for a bar magnet, and as far as actions in the external magnetic field are concerned they will be identical.

The law of direction is seen at once if we remember that lines of force enter a magnet by its south pole, and leave

Fig. 84.



it by its north pole.

Cor. 2. To find the magnetic moment of the helix we have only to remember that for each equivalent magnetic shell (as PQ, fig. 83), if ρ be the density of the magnetic distribution, $\rho \times PQ$ is the strength of the shell, while $nPQi$ is the strength of the equivalent voltaic circuit. Hence $\rho = ni$ and the quantity of magnetism distributed over each pole is $A\rho$ or Ani when A is the area enclosed by the circuit. Hence the magnetic moment, if the length be l, becomes $lAni$. In this expression lnA is clearly the sum of the areas enclosed by all the coils. This may be written A', and we have for the magnetic moment of the electromagnetic helix

$$G = A'i,$$

and the Earth's couple upon it is $HA'i$.

Cor. 3. The result of the preceding Corollary may be used to determine a current in absolute electromagnetic measure.

For let a helix such as that considered be suspended by a bifilar arrangement perpendicular to the plane of the meridian, and let a current be transmitted through it by means of the wires of suspension; then if the Earth's force deflect it through an angle ϕ, the couple upon it will be

$$HA'i \cos \phi,$$

where H is the horizontal component of the Earth's magnetic force.

This must be balanced by the force of torsion in the suspending wires. If D be the coefficient of torsion this is measured by $D \sin \phi$;

$$\therefore \quad HA'i \cos \phi = D \sin \phi;$$

$$\therefore \quad i = \frac{D}{HA'} \tan \phi.$$

Let the current at the same time be passed through a tangent galvanometer in which there is a deflection ϕ'. Then, anticipating Prop. XIII. Cor. 2,

$$i = \frac{H}{\Gamma} \tan \phi'.$$

Multiplying, we have

$$i^2 = \frac{D}{A'\Gamma} \tan \phi \, . \tan \phi';$$

$$\therefore \quad i = \sqrt{\frac{D}{A'\Gamma} \tan \phi \, . \tan \phi'};$$

which gives the current-strength in absolute measure.

Eliminating i, we have

$$H = \sqrt{\frac{\Gamma D}{A'} \tan \phi \, . \cot \phi'},$$

another method of finding in absolute measure the Earth's Horizontal Force.

Cor. 4. If the helix have an iron core its magnetic strength is enormously increased. The strength of the magnetic field within the helix is shown to be $4\pi ni = H$ suppose, and if k be the coefficient of induction for the iron core the density of the separated magnetisms will be Hk. Hence if C be the area of the section of the core, the strength of its pole will be HkC. But the number of lines of force proceeding from a unit pole is obviously 4π. Hence the number of additional lines of force due to the iron core is $4\pi HkC$, and the whole number of lines of force pro-

ceeding from the pole is

$$HA + 4\pi HkC$$
$$= H(A + 4\pi kC)$$
$$= 4\pi ni(A + 4\pi kC).$$

The strength of each pole of the compound magnet due to the helix and its core is

$$ni(A + 4\pi kC),$$

and its magnetic moment is

$$lni(A + 4\pi kC)^*.$$

288. Prop. XII. To find the coefficient of mutual induction for two coaxial solenoids.

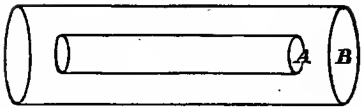
We will call the inner helix A the primary circuit, and the outer B the secondary. Let A, B represent their areas, and n_1, n_2 the number of turns of wire per unit length, i_1, i_2 the currents they carry.

The strength of field anywhere within A

$$= 4\pi n_1 i_1.$$

* This result would show that a magnet of great strength could be made by simply increasing the iron core, and without increasing the number of winds of wire. In practice this is not the case, since the induction of one part of the core upon another so weakens its magnetism that if the section exceed a certain very moderate limit the power of the pole is increased only in a ratio far below that of its increase in section, and the result is a practical weakening of the pole. This is partly obviated when the core consists of a bundle of wires as is usual in induction coils, but it seems probable we may look for greater gain in power by the construction of tubular electromagnets, said to have been first made in Germany by Römershausen in 1850 (though ascribed by Daguin to M. M. Favre and Kunemann). The invention seems to have received only a passing notice till exhibited in the Scientific Loan Collection of the past summer (1876), by Mr Faulkner of Manchester. They are constructed by enclosing an ordinary electromagnet at successive stages during the winding in an iron tube. Each iron tube may be regarded as part of the core removed from the central portion, the induction of the central portion on it being thus weakened. This weakened induction is found more than to counteract the absence (and indeed reversal) of the field due to that portion of the coil which in building up the magnet is placed within the tube, which acquires a strong polarity in the same direction as the core owing to the field of those coils which lie external to it.

Fig. 85.



Hence the whole number of lines of force within A

$$= 4\pi n_1 i_1 A.$$

But B includes all A's lines of force and no more (neglecting the distribution on the ends of A).

Hence the number of lines of force included by B

$$= 4\pi n_1 i_1 A,$$

and the effective strength of the current in B is $ln_2 i_2$, where l is the length of B.

Hence the potential of A on B

$$= 4\pi l n_1 n_2 A i_1 i_2$$
$$= M i_1 i_2,$$

when M is the coefficient of mutual induction.

Hence $$M = 4\pi l n_1 n_2 A.$$

COR. 1. To find the coefficient of self-induction of A, we must make B and A coincide, and divide by 2 the resulting value of M. (Art. 250.)

Hence $$L = 2\pi l n^2 A.$$

COR. 2. If A have an iron core of area C, the number of lines of force enclosed by A becomes

$$4\pi n_1 i_1 (A + 4\pi k C);$$

hence the coefficient of mutual induction

$$M = 4\pi n_1 n_2 l (A + 4\pi k C),$$

and the coefficient of self-induction

$$L = 2\pi n^2 l (A + 4\pi k C).$$

289. *Galvanometers.* These instruments generally consist of a coil of wire carrying the current to be measured, and a magnet needle suspended near the centre of the coil, the movements of the magnet in the electromagnetic field made by the coil giving the means of measuring the current strength.

The first thing we require to determine is the strength of the field at the place where the magnet is suspended, and if the galvanometer is to give absolute measurement, the strength of the field near the needle must be as uniform as possible. In all cases of a magnetic field due to a current, the strength of the field is directly proportional to the current-strength, and knowing the strength of the field with unit current we can at once compute it with any current. This strength of field near the magnet due to unit current is called the 'Galvanometer constant,' and we shall denote it by the symbol Γ.

290. Prop. XIII. To find the Galvanometer constant for a galvanometer consisting of a few turns of wire in the form of a circle the needle being suspended at its centre.

If a be the average radius of the wire, and n the number of turns, the strength of field with a current i is shown (Art. 284) to be

$$\frac{2\pi n i}{a},$$

at the centre of the coil.

Hence if i be unity the strength becomes

$$\frac{2\pi n}{a} = \Gamma.$$

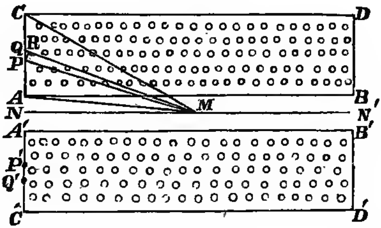
The objection to this form of galvanometer lies in the fact that the field near the centre is not uniform.

291. We have already shown that a sensibly uniform field may be produced by coiling the wire in the form of a long helix. In this case, however, it is impossible to observe the movements of a needle suspended inside it. Arrangements have been made consisting of two or three parallel circles having the needle suspended symmetrically

between them. By this means, originally due to Helmholtz, a field of great uniformity may be produced. To calculate the strength we should use the formula of Art. 284, summing the strengths due to each circle. This condition is also nearly satisfied by the mirror galvanometer considered in the next Article.

292. The coil of Thomson's Mirror Galvanometer consists of a solid cylinder of wire surrounding a central cylinder in which the magnet and mirror swing.

Fig. 86.



Suppose the figure to show a section of the galvanometer and M the position of the mirror and magnet.

The coil $ABCD$, $A'B'C'D'$ may be regarded as a number of coaxial helices, the end of one of which is represented in section by PQ, $P'Q'$. If there be n thicknesses of wire to a unit of length the strength of current in $PQ = nPQ$ due to unit current in the wire.

Hence the strength of field at M due to the elementary helix

$$= 4\pi nPQ \cos\psi, \text{ (Art. 286)}$$

where $\psi = QMN$,

$$= 4\pi nc \cdot \frac{PQ}{QM},$$

where $2c =$ the thickness of the coil.

But $PQ : QR :: QM : QN$;

$$\therefore \frac{PQ}{QM} = \frac{QR}{QN} = \frac{PQ + QR}{QM + QN} = -\log\left(1 - \frac{PQ + QR}{QM + QN}\right)$$

$$= \log(QM + QN) - \log(PM + PN);$$

therefore adding these successive differences, the strength of the field at M or

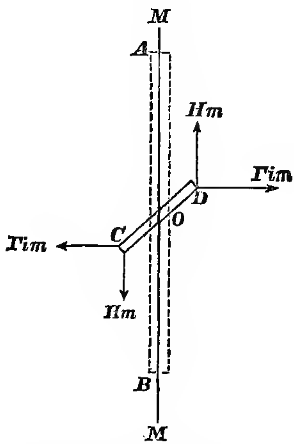
$$\Gamma = 4\pi nc \{\log (CM + CN) - \log (AM + AN)\}$$

$$= 4\pi nc \log \frac{CM + CN}{AM + AN}.$$

COR. 1. If Γ be the galvanometer constant, the strength of the field near the magnet will be Γi when a current i circulates in it, and if we substitute Γi for H the previous propositions about magnet motion are applicable.

COR. 2. If the plane of the galvanometer coil AB be the plane of the meridian the strength of the current will be proportional to the tangent of the deflection of the magnet.

Fig. 87.



Let CD be the magnet, and let the strength of each pole be m. Also let H be the earth's horizontal force.

The forces acting on the poles will be $\pm Hm$ parallel to the magnetic meridian due to the Earth's magnetism, and Γim perpendicular to the meridian due to the voltaic coli.

Hence taking moments about O, we have, if δ be the deviation AOD,

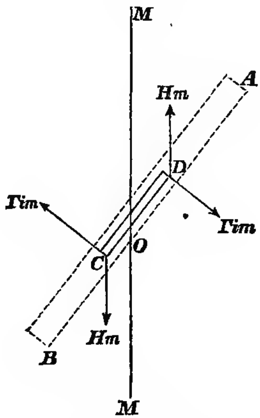
$$\Gamma im \,.\, CD \cos\delta = Hm \,.\, CD \sin\delta;$$

$$\therefore\ i = \frac{H}{\Gamma}\tan\delta.$$

This form of galvanometer is called a Tangent Galvanometer.

COR. 3. If the coil be movable about a vertical axis, and be turned round so that the magnet is in the plane of the coil, the current-strength is proportional to the sine of the deflection. For using the same notation as before and taking moments about O,

Fig. 88.



$$\Gamma im \,.\, CD = Hm \,.\, CD \sin\delta;$$

$$\therefore\ i = \frac{H}{\Gamma}\sin\delta.$$

This form of the instrument is called a Sine Galvanometer.

293. Prop. XIV. To find the throw of a galvanometer needle owing to the passage of an instantaneous electric discharge through it.

If the strength of current at any instant be i, and last for a short interval τ, we have by Art. 273

$$M(\omega' - \omega) = G\Gamma i\tau,$$

M being the moment of inertia of the magnet, Γi the strength of the field, and G the moment of the magnet.

But $i\tau = q$ the quantity transmitted;

$$\therefore\ M(\omega' - \omega) = G\Gamma q.$$

We assume that the discharge takes place so rapidly that the magnet does not move sensibly from its position of rest while the current lasts. Hence the field will during the whole discharge be perpendicular to the magnet. In this case we can add both sides of the last equation during the whole discharge, and we have, if ω_0 be the impulsive angular velocity and Q the quantity transmitted,

$$M\omega_0 = G\Gamma Q.$$

But by Art. 278 if an angular velocity ω_0 be imparted to the magnet, and if α be the throw of the needle,

$$\omega_0 = 2\sin\frac{\alpha}{2}\sqrt{\frac{HG}{M}};$$

$$\therefore\ G\Gamma Q = 2\sin\frac{\alpha}{2}\sqrt{HGM},$$

$$\text{or}\quad Q = 2\frac{H}{\Gamma}\sqrt{\frac{M}{HG}}\cdot\sin\frac{\alpha}{2}.$$

If T be the time of a single vibration of the needle under the Earth's magnetism,

$$T = \pi\sqrt{\frac{M}{HG}};\quad (\text{Art. } 279)$$

$$\therefore\ Q = \frac{2H}{\Gamma}\cdot\frac{T}{\pi}\cdot\sin\frac{\alpha}{2}.$$

If the constants H, Γ, T, are known this equation gives us a means of measuring any electrical accumulation. In practice, however, it is usual to have condensers of known capacity, which can be charged by a battery to a known

poteutial, and then for any other accumulation the quantities will be proportional to the sine of half the angle of throw.

Cor. This method is applicable to measure a total induced current since it lasts but a very short time, and this is all that is assumed in the preceding investigation.

294. Prop. XV. To explain the action of the Dead-beat Galvanometer.

In a galvanometer with the needle swinging inside the coils, the movement of the poles produces an induced current which 'damps' the swing, or, in other words, produces a field, whose action on the needle opposes its movements.

Since the two poles move in exactly opposite directions their separate effects will be simply added. The strength of field produced by the induced current is found to be increased by increasing the number of turns in the galvanometer coil, and if this number be made great enough it may entirely check the free vibration of the magnet about its position of rest after the electromotive force producing the first elongation has sunk to zero (see Art. 252). The consequence of this will be that as the needle returns from its first elongation, the motion is so much damped that it merely returns slowly to its position of rest, never passing it, so that the motion ceases to be one of oscillation.

This form of galvanometer is extremely useful in marine telegraphy, as it would be highly inconvenient to wait for the needle's return to rest between two consecutive signals.

These galvanometers are very expensive, owing to the enormous number of winds required in the wire coil. Their resistance is often as much as 30,000 or 40,000 ohms.

295. Prop. XVI. To find the intensity of the current in Delezenne's circle.

This circle consists of a circular wire rotating about an axis which is fastened to a frame-work, and can be adjusted to any position.

1. Let the axis of rotation be perpendicular to the line of the dip, and let the magnetic intensity be H.

If A be the area of the circle, the number of lines of force included when the plane of the circle is perpendicular to the dip is HA. On turning the circle round its axis the number of lines included decreases, till after a quarter of a revolution it becomes zero, and after half a revolution $-HA$. The total current during the half revolution is measured by $\frac{2HA}{R}$, where R is the resistance of the wire.

From this point the number of lines of force increases again, and the induced current would be in the next half-revolution of the same strength, but in the opposite direction. To obviate this a commutator is arranged, so that when the circle in its revolution comes to this point, the current through the galvanometer is reversed. Hence the total current through it during one revolution becomes $\frac{4HA}{R}$.

If the circle make n revolutions per second, the measure of the quantity transmitted per second, or of the current, is

$$\frac{4nHA}{R}.$$

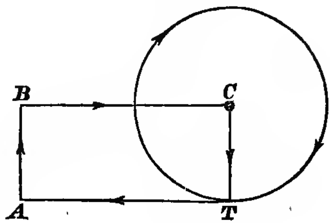
2. Let the axis be inclined to the line of the dip at an angle θ. The only difference will be, that the only part of the Earth's force effective is that perpendicular to the axis of revolution, and we have therefore to write $H \sin \theta$ for H in the preceding result.

Cor. By increasing the number of turns of wire in the circle we increase the number of lines cut through in the same ratio, but we also increase the resistance in the same ratio. R however contains the resistance of the wire coils and of the galvanometer employed, so that if the resistance of the galvanometer be large compared to the resistance of the wire coils, we increase the current by increasing the number of turns; but if the galvanometer resistance be small compared to the resistance of the circle, we do not increase the current by multiplying the turns.

296. Prop. XVII. To explain the action of Thomson's electric current accumulator.

This consists essentially of a circular plate revolving about an axis parallel to lines of magnetic force. The plate at one point makes contact with a fixed spring or mercury cup as T, and the circuit is completed by wires TA, AB, BC, BC being forked so as to make contact with the axis C without interfering with the rotation of the plate.

Fig. 89.



If the plate be rotated in the direction of the arrow, and the lines of force be downwards, the induced current will be in direction $CTAB$ round the closed circuit. The motion of CT clearly opposes the electromagnetic repulsion between the parallel and opposite currents CT, AB. This motion will therefore constantly tend to strengthen the induced current.

Let CT turn through an angle θ in time τ so that $\dfrac{\theta}{\tau} = \omega$, the angular velocity of the plate, and let a be the radius, the area traced out by the moving conductor CT is

$$\frac{1}{2}a^2\theta.$$

The strength of the field is made up of H the magnetic strength, and $\dfrac{2i}{c}$ the strength of the field due to the electromagnetic action of AB, i being the current-strength, and c the distance BC (Art. 282).

Hence the electromotive force in the circuit (Art. 255, Cor. 2)

$$= \frac{1}{2} a^2 \frac{\theta}{\tau} \left(H + \frac{2i}{c}\right)$$

$$= \frac{1}{2} a^2\omega \left(H + \frac{2i}{c}\right).$$

Then, as in Art. 255, the equation for the current will be

$$\frac{1}{2} a^2\omega \left(H + \frac{2i}{c}\right) i\tau = Ri^2\tau + L(i'^2 - i^2),$$

when R is the whole resistance, and i' the current-strength at the end of the small interval τ.

The case of special interest is when $H = 0$, supposing that after the current has reached a certain value i_0 the magnetic field is reduced to zero. For this case

$$\frac{a^2\omega}{c} i^2\tau = Ri^2\tau + 2Li(i' - i);$$

$$\therefore \left(\frac{a^2\omega}{c} - R\right)\tau = 2L\frac{i' - i}{i} = 2L \log \frac{i'}{i}$$

$$= 2L(\log i' - \log i).$$

If i_0 be the initial value, and i the value after a time t, we have on summation

$$\frac{a^2\omega - Rc}{c} . t = 2L \log \frac{i}{i_0},$$

or $$i = i_0 \epsilon^{\frac{a^2\omega - Rc}{2Lc}};$$

which shows that if $\omega > \dfrac{Rc}{a^2}$, the current goes on constantly increasing in compound interest ratio.

297. **Prop. XVIII. To find the value of the velocity which determines the ratio between the different electrical units in electrostatic and electromagnetic measure.**

We have shown in the previous chapter that this ratio is a velocity which will be independent of any system of fundamental units adopted.

Of the numerous methods which have been employed, we give two, the principles of which will be easily understood.

Method 1. To compare directly the charge of a condenser in electrostatic and electromagnetic measure.

Let a condenser be constructed of such material and form, that its capacity can easily be calculated in electrostatic measure. By means of a battery this condenser can be charged to a potential, which can be measured by an electrometer in absolute electrostatic measure. The quantity in electrostatic measure, if C represent the capacity, and V the potential, is given by

$$Q = CV \qquad\qquad (1).$$

Discharge the same condenser through a galvanometer. Then by Art. 293, if $\bar{Q}$ be its charge in electromagnetic measure,

$$\bar{Q} = \frac{2H}{\Gamma} \cdot \frac{\mathrm{T}}{\pi} \cdot \sin\frac{\alpha}{2} \qquad\qquad (2),$$

then by Art. 266, $\dfrac{Q}{\bar{Q}} = v$, and the value of v becomes known.

298. *Method* 2. To compute the value of v in terms of a resistance.

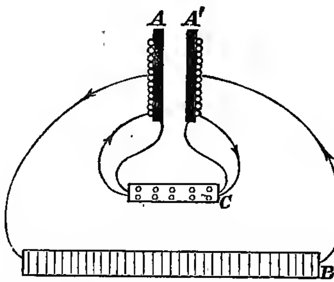
We have shown in Chapter IX. that in electromagnetic measure resistance is of the same order as a velocity, and we defined the ohm as a velocity of 10^9 c.m. per second.

This method, due to Professor Clerk Maxwell, requires the use of a battery of very high electromotive force and a set of high resistances.

Two brass plates are placed so that one is moveable, and are kept at a certain difference of potential; there is in consequence an electrostatic attraction between them. On the back of each of these plates is coiled a wire, so that the battery-current goes in the two wires in opposite directions; there will then be an electromagnetic repulsion between these currents.

The method consists in so adjusting the resistance and distance of the plates that this attraction and repulsion shall balance each other.

Fig. 90.



Let A, A' be the discs, B the battery, and C the large resistance. The current from the battery goes through the two coils on A, A' and through the large resistance in C. Hence if E be the difference of potential at the extremities of C in electromagnetic measure, the current-strength is given by

$$i = \frac{E}{R}.$$

Hence the repulsion between the two discs will be

$$\kappa i^2 \text{ or } \kappa \left(\frac{E}{R}\right)^2,$$

where κ depends on the geometry of the coils, and can only be computed by approximate methods.

To bring the brass plates to a difference of potential, they are connected with the terminals of C. This difference

is then in electromagnetic measure E, and therefore in electrostatic measure $\frac{E}{v}$ (Art. 266).

We have shown (Art. 88) that the attraction between the plates will be

$$\frac{E^2 \cdot a^2}{8v^2b^2},$$

when a = the radius of each plate, and b = the distance between them.

Hence when the adjustments are completed,

$$\frac{E^2 \cdot a^2}{8v^2b^2} = \kappa \frac{E^2}{R^2};$$

$$\therefore \quad v = \frac{a}{2\sqrt{2\kappa} \cdot b} R,$$

which gives v in terms of R, and R being measured in ohms can be at once converted into velocity on multiplying by 10^9.

A complete account of this method, which is due to Prof. Clerk Maxwell, together with the various adjustments required in practice, will be found in *Phil. Trans.* for 1868.

299. The results obtained by these and other methods give numerical results not very discordant, varying between 282 and 310 million metres per second, the mean being nearly 300 million metres per second. The remarkable agreement between this velocity, and the various determinations of the velocity of light (which varies with different observers between 298 and 314 million metres per second), points to an intimate connection between the phenomena of electromagnetism and light. Prof. Clerk Maxwell has developed a theory of light endeavouring to show on mechanical principles that the medium through which electromagnetic actions take place may be identical with the æther which transmits the vibrations of light.

EXAMPLES ON CHAPTER X.

1. Show that the Moment of Inertia of a thin circular wire about an axis through its centre and perpendicular to its plane is Ma^2, where M is its mass and a its radius. Deduce the Moment of Inertia of a broad circular annulus about an axis perpendicular to its plane through its centre.

Ans. $\frac{1}{2} m (a^2 + b^2)$, where $m =$ the mass, and a, b the external and internal radii. (Cf. Chap. I. Ex. 36.)

2. Find the Moment of Inertia of a thin straight bar about an axis through one extremity.

Ans. $\frac{1}{3} m l^2$, where m is the mass and l the length.

3. Find the Moment of Inertia of the same rod about its middle point. Ans. $\frac{1}{12} m l^2$.

4. Show that the oscillations under gravity of a bar freely suspended by its end will be synchronous with those of a fine string two-thirds the length, having a particle at its extremity whose mass is equal to that of the bar.

5. A magnet A is placed so that its axis produced, bisects at right angles the axis of another magnet B, the distance between their centres being great compared to their lengths. Make an approximation to the couple produced by A on B, and that produced by B on A.

Ans. $\dfrac{8aa'mm'}{c^3}$, $\dfrac{4aa'mm'}{c^3}$, where $2a$, $2a'$ are the lengths of A and B, c the distance between their centres, and mm' their magnetisms.

6. A long magnet acts on a small compass-needle placed on its axis. Find the error produced by it on the compass in different directions of the disturbing magnet.

7. A long magnet acts on a small compass-needle placed in the line bisecting its axis at right angles. Find the error produced by it on the compass for different directions of the disturbing magnet.

8. One end of a magnet is prolonged by a thin stem of gumlac which carries a small pith-ball, the other end having

a counterpoise. An equal ball is so fixed that the two are just in contact when the magnet is in the meridian. The two balls are electrified to a potential V, and the magnet is observed to be deflected through an angle 2α; show that V^2 varies nearly as $(\sin \alpha)^3$.

9. A hole is pierced in a card through which passes a straight wire carrying a current. Iron filings are sprinkled over the card, and the card gently tapped. Find the form assumed by the iron filings.

10. If a magnet be placed anywhere in the magnetic field due to a straight current, show that the magnet has no tendency to rotate, as a whole, round the current.

11. Deduce the principles which guide us in experiments on the rotation of a magnet round a current, and a current round a magnet.

12. In the experiment of the last question, show that the whole amount of work spent in each rotation of the magnet pole round the current or vice versa is $4\pi mi$, where m is the strength of the pole and i the current-strength in the conductor, whose length is supposed to be infinite.

13. A magnet is suspended in a horizontal plane so as to be free to move about its south pole, and a vertical current is approached towards it.

(i) The conductor being outside the circle described by the north pole, show that the north pole will be driven by the current to rotate in opposite directions through portions of the circumference bounded by tangents to the circle from the intersection of the conductor with the plane.

(ii) The conductor being within the same circle, the direction of movement of the north pole will be in all parts of the circumference the same.

(iii) The conductor being on the circumference of the circle, show that the rotation will be always in the same direction.

(iv) Show that no permanent rotation of the magnet can be produced by this means.

14. A magnet NS is supported at its middle point, and a conductor carrying a downward current cuts the horizontal plane at O.

(i) A circle is drawn about the triangle ONS, and a diameter drawn through O. From N, S perpendiculars Na, Sb are drawn on to this diameter. Show that in all positions the moment of the forces on the magnet turning its north pole in a direction right-handed to the conductor is

$$\frac{Gi}{ON\,.\,OS}(Sb \pm Na),$$

G being the moment of the magnet, and the plus sign being employed when the perpendiculars fall on the same side of the diameter.

(ii) If the conductor cut the circumference of the circle of which the magnet is a diameter, there is no tendency to rotate the magnet.

(iii) If the conductor be outside the circle, the direction of rotation is governed by that of the more remote pole.

(iv) If the conductor be within the circle, the direction of rotation is governed by that of the nearer pole.

(v) If the conductor be placed on the line bisecting the magnet at right angles, the rotative force will be nil.

(vi) If the field be divided by the circle of which the magnet is the diameter, by the magnetic axis produced, and by a line bisecting it at right angles; show that on crossing any of these lines if on one side the current appear to attract the magnet, on the opposite side it appears to repel it.

(vii) Draw a diagram showing in what positions the conductor appears to attract the magnet, and in what positions it appears to repel it.

15. Show that in measuring a current by a sine galvanometer if the current be stronger than a certain limit, it will be necessary to shunt the current before measuring it.

16. If a tangent galvanometer be arranged so that it can also be used as a sine galvanometer, show that any current producing more than 45^0 deflection in the instrument,

when used as a tangent galvanometer, must be shunted before being measured by it as a sine galvanometer.

17. In Helmholtz's arrangement for a tangent galvanometer, show that the greatest degree of constancy of magnetic field along the axis near the magnet will be when the distance between the coils is equal to the radius of either coil.

18. Show that in the galvanometer of the last question, the galvanometer-constant is given by $\Gamma = \dfrac{32\pi}{5\sqrt{5}\,.\,a}$, where a is the radius of the coil.

19. A finite wire carrying a current is perpendicular to and on one side of an infinite wire also carrying a current. Find the magnitude and direction of the force exerted by the latter upon the former wire.

Ans. $2ii' \log \dfrac{y_1}{y_2}$, where i, i' are the current-strengths, and y_1, y_2 the distances of the ends of the finite from the infinite wire. The direction will be parallel to the current in the infinite wire when the current in the perpendicular wire is away from it.

20. If the length of a helix be forty times its diameter, show that the strength of the magnetic field within it varies about one-thousandth part through $\dfrac{2}{3}$ of its length.

21. A helix A is placed with its axis perpendicular to the meridian, and a short magnet B is suspended at a point on its axis produced, the magnet being deflected from the meridian by a current in the helix. Another magnet C is now placed with its axis along the axis of the helix produced and moved about till B is again in the meridian.

If $2l'$ be the length, and G the moment of C, $2l$ the length, and Ai the moment of A (i being current-strength), a' and a the distances of the middle points of A and C from the suspension of B, then show that

$$\frac{G}{l'}\left\{\frac{1}{(a'-l')^2}-\frac{1}{(a'+l')^2}\right\}=\frac{Ai}{l}\left\{\frac{1}{(a-l)^2}-\frac{1}{(a+l)^2}\right\}.$$

22. In any electromagnetic engine in which the motive power consists in alternate attractions and repulsions of electromagnets, the greatest amount of external work is obtained when, by setting the engine in motion, the current is diminished by one half.

Let m' be the average magnetism of the electromagnets when the machine is in motion:

μ' the mean power of attraction acting on the moving magnet. This will be proportional to m'^2 supposing the same current in both electromagnets:

i' the average current-strength.

Let also m, μ, i denote the same quantities when the engine is at rest:

v be the rate of motion of the electromagnets:

β be the number of turns of wire in the electromagnet which may or may not have an iron core:

a be a constant such that $a\beta$ is the strength of the electromagnetic pole for a unit current.

r be the total resistance in the circuit,

and E be the electromotive force.

Then we have the following relations:

By Art. 287, $$m' = a\beta i' \quad \text{.........(1)},$$

and $$m = a\beta i \quad \text{.........(2)}.$$

By Art. 255, $$i_1 = i - i' = \frac{kv\beta m'}{r} = \frac{kav\beta^2 i'}{r} \quad \text{.........(3)},$$

when k is a constant depending on the particular form of engine.

By Ohm's Law $$i = \frac{E}{r} \quad \text{.........(4)}.$$

By (3) and (4) $$i' = \frac{E}{r + kav\beta^2}.$$

Hence $$m' = \frac{aE\beta}{r + kav\beta^2} \quad \text{and} \quad v = \frac{aE\beta + m'v}{kam'\beta^2}.$$

Again, the work obtained from the engine per second will be measured by $\mu' v$, or

$$W = \mu' v = Cm'^2 v,$$

or $$\frac{W}{C} = \left(\frac{aE\beta}{r + kav\beta^2}\right)^2 v = \frac{(aE\beta - m'r)\, m'}{ka\beta^2},$$

it is easily seen by Algebra that the denominator of the first expression is the least possible when

$$v = \frac{r}{ka\beta^2},$$

and that the numerator of the second is the greatest possible when

$$m' = \frac{aE\beta}{2r};$$

these both correspond to the greatest possible value of W, the work done.

But when the machine is at rest,

$$m = a\beta i \quad \text{by (2)}$$
$$= \frac{aE\beta}{r} \quad \text{by (4)};$$

$$\therefore \quad m' = \tfrac{1}{2} m, \qquad \text{and} \quad i' = \tfrac{1}{2} i,$$

or the work is the greatest possible when the current and magnetism of the electromagnets are reduced by the motion to half the corresponding quantities when the machine is at rest. (Wiedemann.)

23. If an amount of work W is obtained from any electro-magnetic arrangement without a falling off of current-strength, show that the battery must yield an extra amount of energy measured by $2W$.

24. Deduce the propositions of Arts. 283—286 by substituting for each simple voltaic circle a plane magnetic shell, and applying to the magnetisms on its opposite faces the propositions of Art. 31 and Chap. II. Ex. 7.

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